Getting the Measure of the World

Calendars, lengths and mathematics

Everything is mathematical

Getting the Measure of the World

Getting the Measure of the World

Calendars, lengths and mathematics

Iolanda Guevara and Carles Puig

Everything is mathematical

Published by RBA Coleccionables, S.A.
c/o Hothouse Developments Ltd
91 Brick Lane, London, E1 6QL

Localisation: Windmill Books Ltd.
Photography: iStockphoto

ISSN: 2050-649X

Printed in Spain

Contents

Preface

"I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind." This sentence is attributed to the Scottish physicist and mathematician William Thomson (1824-1907), also know as Lord Kelvin, who created the scale of absolute temperatures in 1848. In his honour, the unit of temperature in the International System of Units bears the name 'kelvin'.

Undoubtedly, as Thomson said, measuring is a fundamental part of modern science, but it is also an essential activity in daily life. Knowledge and command of even our most immediate surroundings would not be possible without carrying out some kind of measurement. The most sensitive, most reliable and most precise measurements require the use of mathematics.

Throughout history, human beings have developed different strategies for measuring. Those strategies have enabled them to measure the planet on which they live, interstellar and intergalactic space, and time. They created appropriate units for those measurements and understood that measurements can be calculated both directly and indirectly. Since ancient times humankind has been aware that natural cycles were important for the existence of life, and soon discovered that time could be measured through observation of the skies and the motions of the heavenly bodies. They related those motions to events in their natural environment – cold, heat, the falling of leaves, the birds migrating and so on. The natural surroundings were also measured, first locally – so as to form boundaries for their inhabited lands – and then further afield in order to plan longer journeys and trading expeditions. Increases in trade exposed the existence of numerous local units of measurement, and the difficulties they caused gradually became more evident. Some universal standards had to be found. The first universal standard, the metre – which gave rise to the decimal metric system – was set following a precise measurement of the Earth that required a very particular mathematical technique known as triangulation. That method and other mathematical techniques related to astronomical, geodetic, calendrical and metrological measurements are the subject of this book.

The act of measuring is in fact a day-to-day activity. In one way or another, all of us are constantly carrying out measurements. For instance, we get up in the morning and first of all we might 'measure' out some milk to put on our cereal

or to make some tea. If we counted up all the daily activities that involve some sort of measurement the total number would be astonishing.

We would like to thank the publishers at RBA for the trust they placed in us and for enabling us to take part in this series, together with their willingness to accept the initial project on the history of the calendar and the introduction of the metre. Thanks to their suggestions, the initial project eventually turned into a more wide-ranging book, covering different aspects of measurement. We must express our gratitude to Roser Puig for providing advice on some technical details concerning the Islamic calendar. We would also like to thank Anton Aubanell Pou for encouraging us to write this book, for offering his enthusiastic and unconditional support and for making available his own works and documentation on the history of the calendar and the birth of the metre.

Chapter 1

What is Measuring?

Right from birth we begin a long learning process. From a very early age and long before it can utter its first words or take its first steps, a baby develops the ability to determine how near or how far away its mother is and it learns to stretch out its arm just the right distance towards the object it wants to take hold of. It also perceives the differences between similar objects by becoming aware that there are some objects *larger* or *smaller* than others. The fact is that we human beings learn very quickly and in a natural way to find our place in our surroundings by comparing distances, sizes, volumes and so on. In other words, from the time of our birth we learn to measure; but what exactly is measuring?

A ubiquitous activity

An infinite number of daily actions demonstrate just how we humans spontaneously carry out activities involving measurements. Some, such as those outlined below, are situations we are all familiar with.

We walk into the home of some friends; we make our way along the hall and go through the doorway into the dining room; we do it automatically and without hesitating. We are aware that we are 'going through the doorway' and, if we are very tall, we automatically stoop a little.

We have to cross a quiet road and see, in the distance, a vehicle approaching. We decide to cross and we do it calmly, as we have calculated that the vehicle is going to take longer to reach the place where we are than we will take to reach the other pavement.

We write a letter to a friend and fill a page with words; when we finish we take the envelope and realise at once that we have to fold the page in half or fold it twice to make it fit into the envelope.

These examples show how we are constantly making decisions that require a process of comparison which we have 'automated', between magnitudes of the same type: the height of a doorway and our own height, the times taken by two

moving objects (the vehicle and the person crossing the road) so as to cross the road while avoiding a collision; the surface area of an unfolded page and that of an envelope.

Mathematics common to all cultures

Maths is a cultural product just like any other form of knowledge, says Alan J. Bishop in *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*. This mathematical culture shows up in three fields. The first one is related to numbers and involves the practices of counting and measuring; the second refers to space and shows up in the actions of localising and designing; finally, the third applies to people's social relationships and affects the field of explaining and playing. Acknowledging that mathematics forms part of all cultures is a way of saying that 'the world is mathematical'. But what do we understand by mathematics?

Our aim is to attempt to identify human activities that are common to different cultures and that have a relationship with mathematical thinking. In this context we understand that mathematics is not contents, not questions to be studied, but reasoning and mental processes that we put into effect while we are carrying out certain activities with mathematical connotations. At an academic level, we study mathematics at school, college or university, but if we only looked at this field we would be dealing with regulated maths, the maths in the books. Our intention is to go further, to widen the field of study and also look at activities carried out in daily life, outside school, here and in other cultures, which have characteristics in common and which can be said to be mathematics. Language was born from the need to communicate, but where did mathematics come from? What were the needs that had to be met and that created the six activities mentioned above?

How elements are arranged in space is of particular importance in the development of mathematical ideas. *Localising* and *designing* are the two activities that share this task. The former responds to the need to go out in search of food and return home without getting lost. It implies knowing one's surroundings, positioning oneself and getting one's bearings. Three kinds of space can be considered: physical space (that of objects), the socio-geographic space of our surroundings, and the cosmological space of the world we live in.

Designing concerns the creation and manufacture of objects and utensils. These might be destined for domestic use, commerce, ornamentation, warfare or religious

rites. Why do all cultures design bowls for eating liquid food? The answer is quite simple: it is impossible for a flat or convex plate to hold the liquid. Design is also related to the arrangement and structuring of larger spaces, like houses, villages, gardens, roads and cities.

As well as being linked to our physical surroundings, we need to relate to other members of our community. This need to socialise gives rise to other activities related to mathematical thinking: *playing* and *explaining*.

The first of these refers to the rules and social procedures of conduct as well as to the imagination, to posing hypotheses or formulating questions of the type, "What would happen if...?" when a situation is being analysed. All cultures play and, what is more important, they take their games very seriously! This point suggests that play, the recreational activity par excellence, may be closer to mathematics than might be expected. In actual fact, many mathematicians believe that the activity of

ROUND HOMES

Why do we find round houses in several different cultures? Of all the rectangles – or quadrilaterals – that can be built with the same length of perimeter, a square is the one that has the largest internal area. However, of all figures with an equal perimeter the one with the largest internal area is a circle. Round homes must therefore be the most economical to build because they require the least material (of bricks, ice, reeds, animal skins or other materials) to hold a living area. There are numerous examples of this reasoning in very different cultures: Inuit (Eskimos), Native American, Central African and others, as can easily be seen by looking at their traditional buildings.

An Inuit igloo, Canada.

A Native American teepee, North American plains.

A Kikuyu hut, Kenya.

play implies a process of reflection similar to that of when we are trying to solve a problem: analysing the situation, seeking strategies, comparing them, choosing the best, carrying it out and checking to see if it works.

Explaining also has more to do with the social environment than with the physical, despite the fact that it relates the two. It is a case of sharing analysis and the conceptualisation of the environment with other members of the community. Without the environment there is nothing to explain, but neither is there if we have no need to share the results of our research. Explaining is seeking the reasons for phenomena, for the similarities or differences, the connections between them and also, therefore, classification. For more complex phenomena an explanation becomes description, in which, from the mathematical point of view, the most noteworthy is the use of 'logical connectors' that enable propositions to be

THE ISHANGO BONE

The Ishango Bone is the fibula of a baboon that bears three sets of marks. The bone was discovered in 1960 by the Belgian archeologist Jean de Heinzelin in Ishango, near the source of the River Nile. The people living some 20,000 years ago near what is now Lake Edward, between the Democratic Republic of the Congo and Uganda, may have been one of the first societies that carried out counting operations. Different studies of the marks and how they are divided into columns have led to claims that perhaps the bone is a tool showing a numerical system. The bone is marked with a series of pairs of numbers, each of which is double the previous one (5, 10; 4, 8; 3, 6), and series of odd numbers (19, 17, 13, 11; 9, 19, 21, 11) in which one of them is the prime numbers between 10 and 20. Finally, it can be seen that the sum of two of the series is 60, and the other one, 48. Some interpretations have linked the markings to calendars; it has been

combined, opposed, extended, restricted, exemplified and elaborated. These processes are the same as those we demand from a mathematical proposition – consistency, elegance and certainty.

Counting and measuring

Counting and *measuring* are the two activities that we most easily imagine related to mathematics because they use numbers. The history of the symbols that different cultures have used for expressing numbers, i.e. the digits or figures, tell us about the needs that our ancestors had to meet and how civilisation provided itself with an indispensable tool to submit their activities to some kind of order. Enter numbers. Georges Ifrah, in *The Universal History of Numbers*, sets out the result of a long process

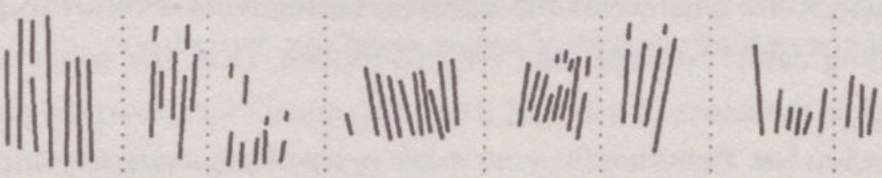
---

claimed that it could be a six-month lunar calendar, and even that it was made by a woman who was investigating the link between the moon's phases and her menstrual cycle.

Left series:

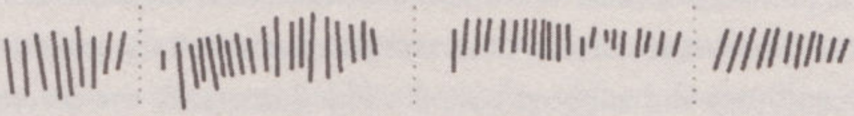
19 + 17 + 13 + 11 = 60

Central series:



7 + 5 + 4 + 1? + 1 + 9 + 8 + 4 + 6 + 3 = 48
5? 10

Right series:



9 + 19 + 21 + 11 = 60

of research into the origin and meaning of numbers in the many and varied cultures that make up the complex web of humanity, and of numerical and calculation systems from antiquity up to modern times.

Through the act of counting, of associating objects to numbers, humanity discovered and described the quantitative aspects of the surrounding world. Throughout history, different cultures and social groups have all needed to count. They counted the days of the year, no doubt because they were aware of the periodicity of the seasonal changes and needed to decide the best time for sowing crops. They counted the members of the community, along with births and deaths. Assets and cattle were counted. On returning from the pastures, the shepherds needed to check if any of the animals were missing.

Counting people, objects and the passing of time were needs that appeared in the beginnings of the history of humankind. Initially, people did not know how to count, such as it is understood today. They were, however, aware of the unit, pairs and the group. Various studies assert that at first glance we can only perceive the numbers from 1 to 4. Above that number it is necessary to count; irrespective of how it is done, it is still counting. This quantification can be carried out in different ways, by counting the elements of each group in question, or by some kind of comparison or mental grouping.

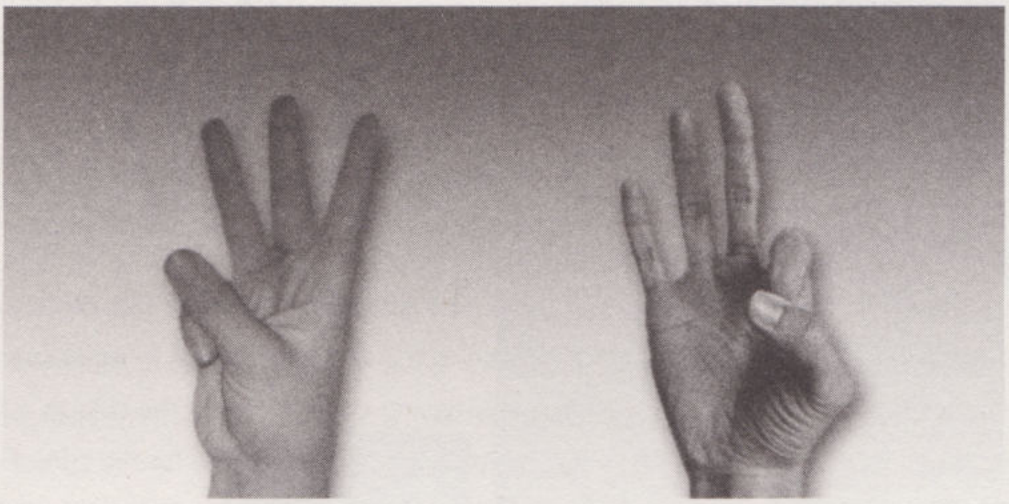
To count – to be able to communicate the date of a ceremony or check that the same number of sheep, goats or cows that were sent out in the morning returned in the evening – several different processes were used.

To memorise the count and communicate it to others, language had to develop, that is, there had to be a name for the number. Humankind took its inspiration from surrounding nature to obtain the cardinal models. Thus, the wings of a bird symbolised a pair; the leaves of a clover, three; an animal's feet, four; the fingers of a hand, five, and so on. There are likewise other types of sequence relationships to arrive gradually at the abstraction of numbers and calculation.

Just as most of us began to count up to ten on the fingers of both hands, most numeration systems that exist nowadays are based on the number ten. Some systems chose base 12, probably because it worked a lot better for divisions as it has more divisors than 10. The Mayas, Aztecs, Celts and Basques added the number of fingers to the number of toes and adopted base 20. The Sumerians, the inventors of the oldest known form of writing, and the Babylonians, the inventors of zero, gave us base 60, which we still use today for dividing the hour into minutes and seconds,

as well as the 360 degrees of a circle and, in turn, each degree into 60 minutes and each minute into 60 seconds.

Various bones of animals with vertical markings or notches discovered at sites in western Europe show how our ancestors counted. This numeration technique is thought to have been the origin of Roman numbers. Another means for counting is by hand. There are indications that show that all the world's peoples used their hands for counting at one time or another in their history. The phalanges and other joints of the hand allowed Egyptians, Romans, Arabs and Persians, as well as Christian peoples of the mediaeval West, to specify the numbers from 1 to 9,999 by following a procedure similar to sign language. The Chinese went further. They thought up a system that allowed them to count up to a hundred thousand on one hand and up to ten billion on both.

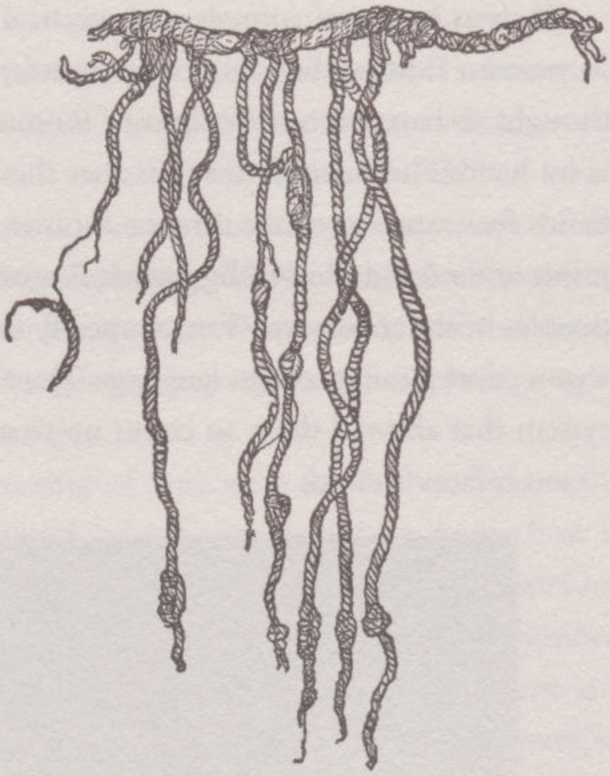


On the left, one of the ways to indicate number three with the fingers. On the right, the Chinese method of expressing the same number.

Another system employed in the history of arithmetic and calculating was using a pile of stones or pebbles, as some writers point out. This method is thought to have been the origin of the abacus. There are still abacus counters in modern-day China, Japan and eastern European countries. The word *calculation* itself is a reference to that method, as the Latin term *calculus* means 'small pebble'.

The writing of numbers, the figures, arrived later. Apparently, some accountants decided to replace the usual pebbles with clay objects. Depending on their shapes and sizes, they represented different quantities in the numeration system: a small stick or rod was the unit; a ball was ten; a larger ball, a hundred, and so on. Archaeological

COUNTING WITH STRINGS: QUIPUS



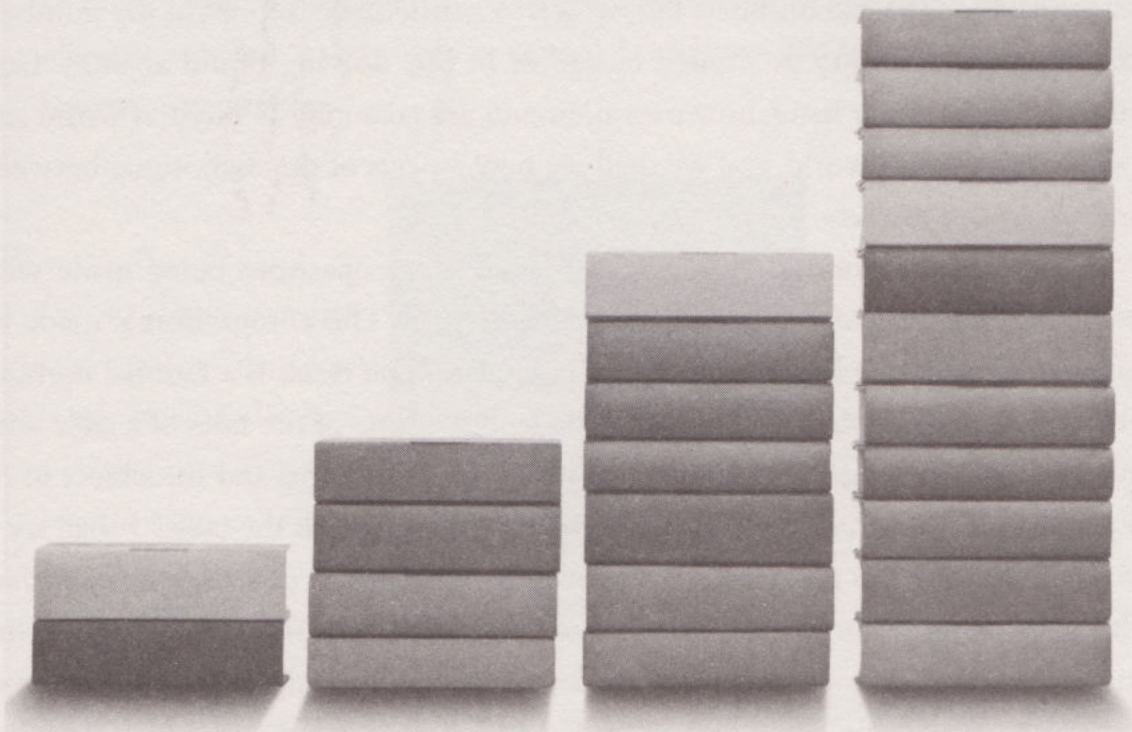
Quipus (from the Quechua language: *khipu*, 'knot') are a system developed by the ancient Andean civilisations who used strings of wool or cotton and knots of one or several colours. The learned men of the Incan Empire, the quipucamayocs (*khipu kamayuq*), used the system for counting. Some researchers claim that it was also used for writing. Quipus have been found in Caral, an archeological site considered by UNESCO to be the oldest city in America (approximately 5,000 years old) situated in the Supe Valley, about 200 km north of Lima, Peru, and also in the focal points of the Wari (Huari) culture, an ancient Andean civilisation that flourished in the central parts of the Andes from around the 7th to the 13th century AD.

A quipu consists of a main string with no knots in it and, hanging from it, other strings of different colours, shapes and sizes and usually having knots in them. The colours refer to different sectors (brown for government; crimson for the Inca; purple for the curaca – the political chief; green for conquest; red for warriors; black for time; yellow for gold; white for silver) and the knots refer to quantities.

traces of this system have been found for a similar period in the 4th millennium BC in two different civilisations – one that counted in base 10, in Elam, an area in Iran near the Persian Gulf, and another in Sumeria, Lower Mesopotamia, which used base 60. The accountants would enclose the objects symbolising the numbers in clay balls. They functioned as small archives and when the day came that the accounts were needed, the clay ball would be broken open and the objects representing the quantity would appear. The system evolved in such a way that little by little the objects placed inside the clay balls were replaced by markings made on the ball itself. A ball became a small notch; a large cone became a wide notch; and a sphere,

became a circle. Thus, around 3200 BC, the first Sumerian written figures appeared, the oldest of any known.

Measuring is the other universally significant activity in the development of mathematical ideas, dealing with comparing, ordering and quantifying. Although there are certain aspects of measurement that all cultures recognise as important, not all use the same criteria when measuring nor do they use the same measurements. Each local environment and each context create necessities which lead to the adoption of one type of measurement or another. For instance, the human body was, in all probability, the first measuring tool used in all cultures. Even today, if there is no tape measure or other exact tool available we measure short distances with strides, and shorter lengths with hand spans.



Measuring implies comparing.

Measuring distances and calculating quantities of foodstuffs were no doubt among the most important needs. In many cultures distances to a place were measured by taking into account the means of travelling there along with the time needed, for instance, days on foot, on horseback or by cart. We still today continue measuring hikes on the moors, for instance, in relation to time, the expected number of hours spent walking. For food, measurements were expressed in relation to the utensils

used for storing them – baskets, cups of rice, sacks and so on. These units still live on. When we are going to cook rice for four people, do we use the scales to weigh it, or do we simply measure it out in cupfuls?

Continuous or discrete

The difference between counting and measuring leads us, in mathematics, to refer to the concept of *continuous* and *discrete*, which can be likened to the continuous and discrete of the physical world when we think of counting sheep or measuring water. Sheep can be individualised; water is a continuous fluid which can be measured, but not counted. In mathematical terms, counting is an activity carried out with whole numbers, or at the most fractions, or rational numbers ($\mathbb{Q}$), while for measuring we need real numbers ($\mathbb{R}$), which in mathematics represent the numbers that depict a continuity as alluded to earlier in our flowing liquid analogy. Let's consider, on the other hand, how measurements are taken in the physical world and in the mathematical world, and we shall see new aspects of the dichotomy between continuous and discrete.

In the physical world, measurement is taken by comparisons being made with a standard that we adopt as the unit of measurement. The comparison is made by multiples or submultiples (fractions) of the standard. The result is a rational number. Let's look at an example. Try to measure the length of one of the sides of a table with a pencil, however long the pencil is. The pencil is the standard and the object to be measured is the length of the table. How many pencils long is the table? I shall try it out while writing this paragraph. The answer is more than 7 pencils, but fewer than 8, in other words, a number between 7 and 8. We shall need fractions to express this result. Now it is a case of comparing the length from where the seventh pencil ends to the end of the table. What fraction of the pencil does it represent? Half? A third? A quarter? This empirical reasoning, reckoning by sight, takes us back to the ancient Egyptians who worked with fractions of numerator 1, except for the fraction 2/3. When measuring the table, if, for example, we take an estimate by sight and see that the table ends at a quarter from the end of the pencil, we would say that it measures 7 3/4. If, on the other hand, we want to be more precise, we can go into the proportion theory of classical Greece, put the measurement down on paper and then apply the Thales theorem until we get a good approximation of the fraction that is nearest to the length we are measuring. In this way, the result in my case was 7 and 2/3.

The result of taking a measurement in daily life is expressed with a fraction or with a finite decimal depending on the procedure and the instrument used for taking the measurement; in any case, it is a rational number. In the example of the table, the result was given in the form of a fraction, 7 and 2/3, by using a pencil as a unit of measurement; when we measured it with a tape measure the result was 1.40 m., a finite decimal. In real life, the act of measuring is an approximation that is dependent on the object being measured, the type of instrument used and the accuracy used in taking the measurement.

REAL NUMBERS

Real numbers (denoted by $\mathbb{R}$) are those that include both the rational numbers (denoted by $\mathbb{Q}$; positive fractions, negative ones and zero) and the irrational numbers (transcendental, algebraic numbers), which cannot be expressed as fractions and have infinite non-periodic decimals, such as: $\sqrt{2}$, π,...

Rational numbers: $-3/4$, $5/8$, $31/7$
Whole numbers: $-7, -1, 0, 5, 20$
Irrational numbers: $\sqrt{2}, (1+\sqrt{5})/2$
Transcendental numbers: $e, \pi, \ln(2)$

Examples of different real numbers ($\mathbb{R}$).

From the natural numbers ($\mathbb{N}$), 1, 2, 3,..., which we use for counting, to the real numbers ($\mathbb{R}$), which we need for measuring in the mathematical model, the extension of the numerical sets can be explained by the need to have numbers to express the result of certain operations:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$
$$(-) \rightarrow (:) \rightarrow \sqrt{\ } \rightarrow \sqrt{(-)}.$$

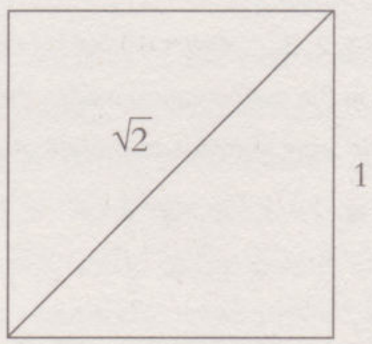
The whole numbers ($\mathbb{Z}$) enable us to express $3 - 4 = -1$; the rationals ($\mathbb{Q}$) enable $\frac{3}{4} = 0.75$ to be expressed; the real numbers ($\mathbb{R}$) express $\sqrt{2}$; the complex numbers express ($\mathbb{C}$), $\sqrt{-4}$.

An exact measurement only occurs in the mathematical model. In this situation, the measurement is continuous, as opposed to discrete. What and how do mathematicians measure? In the history of mathematics, measuring is closely related to geometry, the branch of mathematics that deals with the study of the properties of figures or objects on the plane and in space. It is interesting to note that the origins of geometry go back to the solving of specific problems relating to measurements.

In elementary geometry, objects and shapes are described both in a qualitative and a general way; afterwards, if a more exact and concrete description is needed, it is necessary to quantify; that is where measurement comes in and, to express the measurement, numbers are used. Segments have length; planes have area and bodies in space have volume.

In the mathematical model, measurement is continuous. This statement is borne out by the fact that rational numbers are not enough to express the result of a measurement. We need to extend the set to include all the numbers that cover a straight line – the real numbers again. In day-to-day life one of the properties we often measure is length. In the mathematical model we imagine length on a straight line and establish a correspondence between the points on the line and the real numbers representing them.

In the mathematical model we need to use the whole set of real numbers even in cases that are apparently simple. The Pythagoreans discovered that there are incommensurable magnitudes from this simple question: how long is the diagonal of a square with a side 1 unit long? According to the Pythagoras theorem we can reply that the diagonal is $\sqrt{2}$, but the result of this operation, the square root of 2, cannot be expressed by a rational number ($\mathbb{Q}$); instead, we need the irrational numbers – hence we arrive at $\mathbb{R}$.



A square of side 1 unit and diagonal $\sqrt{2}$, as by Pythagoras' theorem it holds that $\sqrt{1^2+1^2}=\sqrt{2}$.

Our Greek predecessors, who only worked with rational numbers in calculation, posed themselves this problem: how can we measure the diagonal of a square if we have no number to express the result? This question led them to talk of measurable magnitudes – those that could be measured by comparison by means of multiples or submultiples of the initial unit – and incommensurable magnitudes, those that could not be measured just by using comparison of fractions or proportions, as in the unit square and its diagonal.

In Book V of the *Elements*, Euclid (c.325–c.265 BC) used his theory of proportion applied to commensurable and incommensurable magnitudes to solve the problem in some way by establishing rules for operating with all types of magnitudes, whether commensurable or incommensurable.

Magnitudes and units

Etymologically, the verb 'measure' comes from the Latin term *metīri* and, according to the dictionary, means "To compare a quantity with its respective unit in order to discover how many times the latter is contained in the former". It also has other meanings, such as "to have a determined size, be of a certain height, length, surface, volume, etc.".

We understand 'measuring' or 'measurement' as the action and the effect of measuring, though 'measurement' is also used to indicate the 'expression of the result of measuring'. Thus, we can say that the measurement of the surface area of our living space is 110 m^2, or that we have measured the temperature of the living room and obtained a measurement of 21.5 °C. Likewise, the word 'measurement' is used in the sense of "each of the units used to measure lengths, areas or volumes". We could, therefore, refer to a 'pint' by saying that it is *a measurement* of capacity that varies depending on the country and whether it is for liquids or for dry goods.

Measuring presupposes a process of abstraction in which the property or characteristic of an object that we want to emphasise is picked out as the element to be quantified, that is, associated with a number. In the case of a book, we might be interested in its height or width if our intention is to place it on a certain shelf, but if what we want is to have an idea of whether it will be useful for pressing flowers that we are going to preserve then we will no doubt attach more importance to its weight.

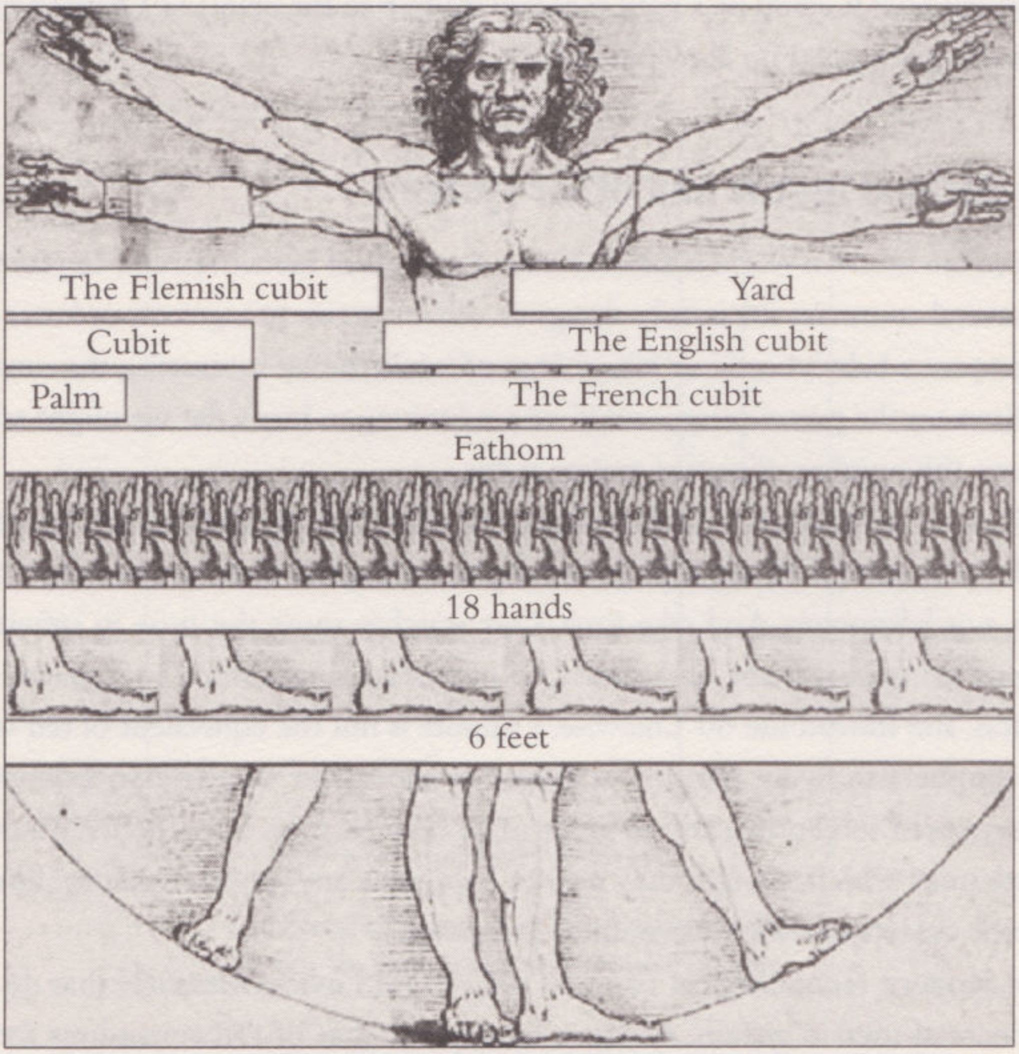
It will help to clarify the concept of *magnitude.* Although the first definition of magnitude in the dictionary is that of "size of a body", we are more interested in using another one: "A physical property that can be measured", as it is closer to what is important to us here. More clear is the definition for quantity given by the International Bureau for Weights and Measures, which defines it as: "An attribute of a phenomenon; a body or substance that may be distinguished qualitatively and determined quantitatively." The process of measuring implies picking out a determined quantity and comparing it, because measuring is comparing an unknown amount that we want to determine and a known amount of the same quantity, which we have chosen to be the unit. In the process of measurement we determine the proportion between the dimension of an object and a specific unit of measurement.

To carry out any measurement it is essential to have a suitable unit available, an amount that is taken as a measure or term for comparison with others of its kind. The result of a measurement is an amount expressed as a number accompanied by the name or symbol of the corresponding unit: 25 kg, 30 m, 28 s and so on.

The results of our need to measure – for planning a journey, buying, selling and then collecting taxes, for example – was that throughout history traditional systems have relied on a large number of different units. In these old systems, the standards were frequently based on parts of the body, on activities related to farming or craft work, or simply by convention within a determined social group.

Plutarch's famous quote "Man is the measure of all things" is apt; since ancient times man has equipped himself with a series of units of measurement that were related to himself, to his own body. Measures that reference parts of the body are now called anthropomorphic units. They include the foot, the handful and the stride. Of course, measures of this type and with the same name frequently differed from era to era and place to place, and so they generally represented a range of different magnitudes.

Long distances were measured in units of time – days on foot or on horseback, an hour's walk and so on. These units are known as itinerary measures. Others were adopted later, such as the stadium, the furlong, the league, the mile, the knot. The mile, for instance, was an itinerary measure that was already in use when Roman roads were being built and was the equivalent of 8 stadia, or 1,000 strides of 5 Roman feet (approximately, 1,375 m in today's units). The Imperial mile introduced in Britain, but largely a U.S. unit today is the equivalent of 1,609 m. At sea, the nautical mile is still used, which measures 1,852 metres.



Anthropomorphic measures.

For measuring land, measures linked to work were traditionally used, such as the 'journey', which was the time needed to work a determined area of farmland.

Measures for the surface area of agricultural land were given as the land's productivity, taking into account the quantity of grain that it could produce. They were, therefore, units that depended on numerous external factors (not least the climate) and were far from invariable.

Traditionally, grain was measured by volume and the unit was its common container, for example, a barrel. This type of measure could cause problems derived from the fact that the same instrument for measuring could have different

equivalences, as it could be a level measure (just up to the brim) or a heaped measure (with the grain piled up above the brim).

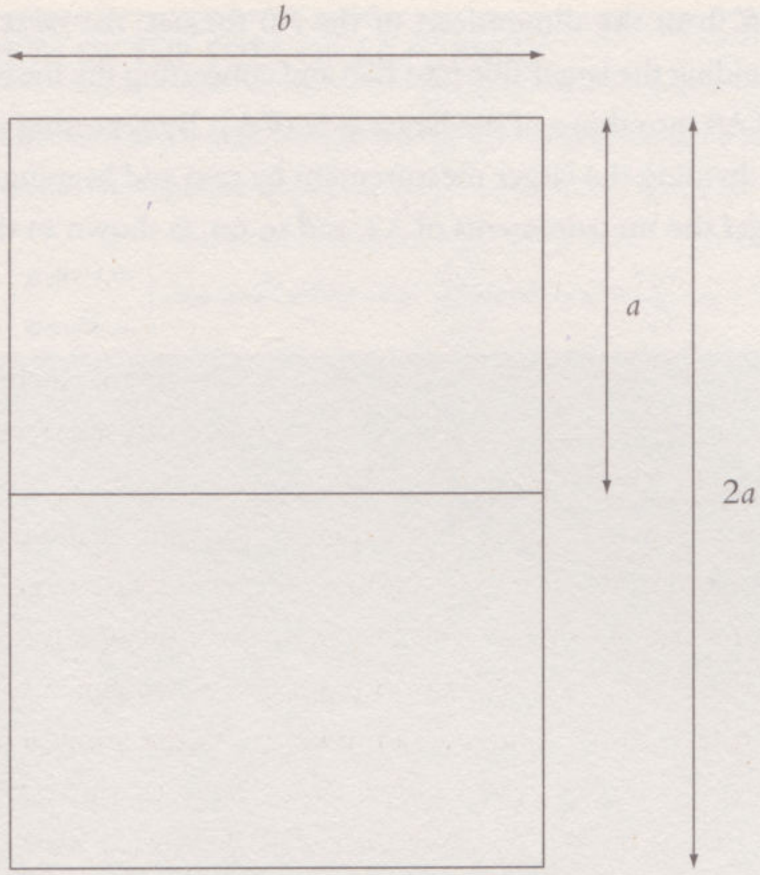
The decimal metric and other systems

Most of the modern world is familiar with the decimal metric system, despite the fact that several countries use it only alongside other systems to a greater or lesser extent. We prepare a baby's bottle or take a dose of medicine by measuring the number of millilitres on the prescription, and if we are looking to buy a flat we might take into account the number of square metres it has.

That, however, does not mean we do not also use units that are not derived from the decimal system. In the UK, for example, we express distances between cities in miles, not kilometres. And take time. Our watches show the time in minutes, but we give no special name and attach no particular significance to a period of ten minutes, and instead use 60. Likewise, a minute is not the equivalent of ten seconds. The numbers showing the size of gloves or shoes, just to give two examples, are not expressed in centimetres or other units derived from a metre. Even nowadays we use units which, though they do not belong to any specific system, allow us to describe aspects of reality we want to point out.

In modern technological contexts we can find useful measures that do not fit the decimal metric system. A classic example is that of DIN measures for paper formats. Although there are other formats for paper, such as the folio, the quarto, or the twelvemo, the most widespread in common use is DIN A4 (210 mm × 297 mm). The standard paper formats adopted by most of the world are based on the German standards which the Deutsches Institut für Normung (German Institute for Standardisation) defined in 1922. The DIN norm was later adopted by the ISO (International Organization for Standardization) standards. Most photocopiers and digital printers, both for home and industrial use, are designed for using DIN format paper.

Three conditions were set for this type of format: the first was that the different sizes had to have the same proportion between their larger and smaller sides; the second was that the successive sizes must have twice the surface area of the following one, so that a sheet of paper of one format could be cut and produce two equal ones of the next format; and the third and last condition was that the largest format, called A0, must have a surface area of 1 m^2.



A format of paper which when folded in half keeps the same proportion between its sides.

How can we get this proportion? If we start off with a format of a rectangular sheet of paper with sides *a* and *b* respectively, the larger format will have 2*a* by *b*. In order for the proportion between its sides to be the same it will have to hold true that:

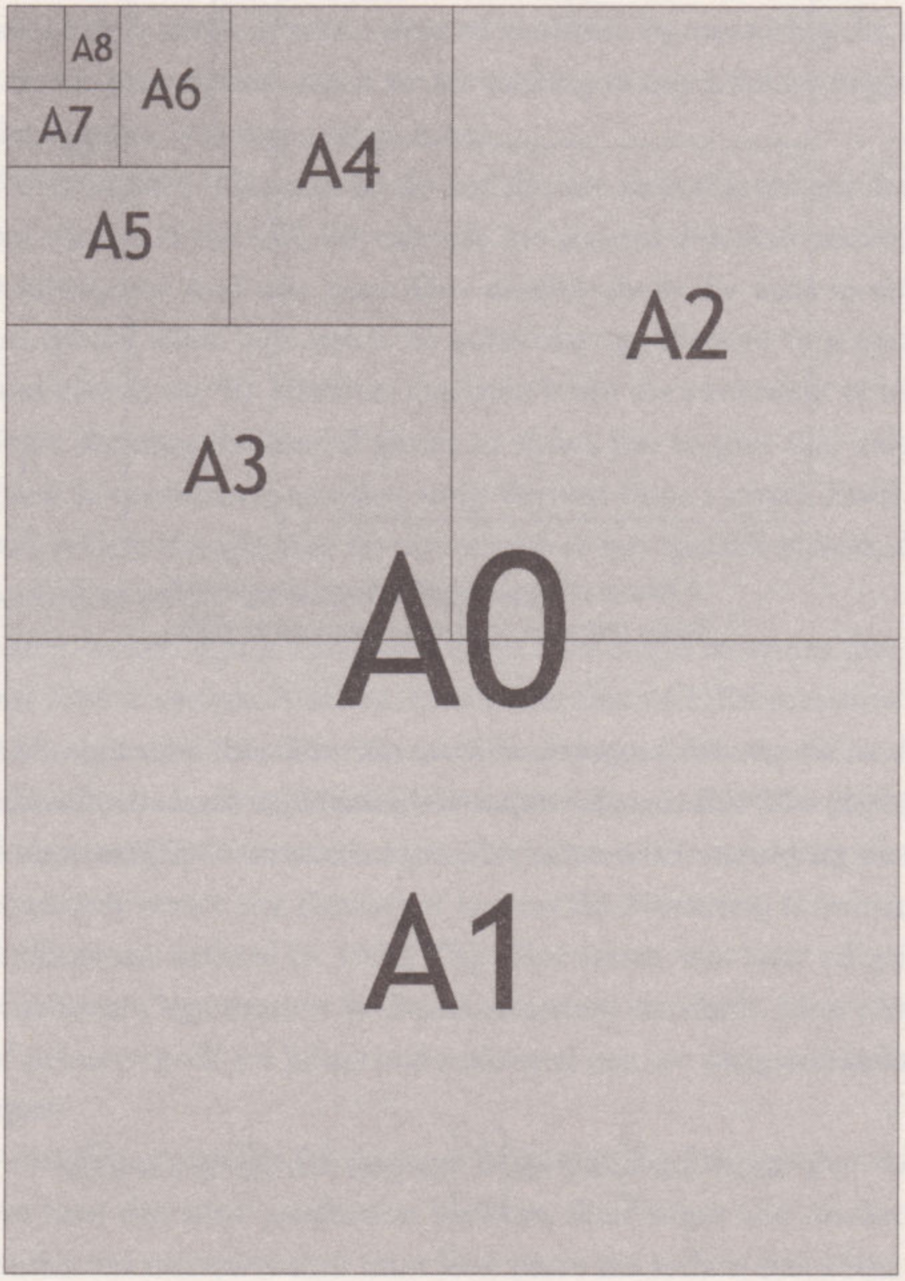
$$\frac{b}{a} = \frac{2a}{b}$$

and therefore:

$$b^2 = 2a^2 \Rightarrow \frac{b^2}{a^2} = 2 \Rightarrow \left(\frac{b}{a}\right)^2 = 2 \Rightarrow \frac{b}{a} = \sqrt{2} \Rightarrow b = \sqrt{2}\, a.$$

In other words, the ratio between the larger side and the smaller one has to be $\sqrt{2}$. If we have a format of paper that fulfills this condition, when it is cut in half it will have the same proportions.

Starting off from the dimensions of the A0 format, the next format (A1) is obtained by dividing the larger side into two and converting the measurements of the smaller side of A0 into those of the larger side of A1. By repeating the process with A1, that is, by dividing the larger measurement by two and keeping the smaller one the same, we get the measurements of A2, and so on, as shown in the figure below:



DIN format sizes.

CALCULATION OF THE SIZES OF A0 FORMAT

A rectangle with sides *a* and *b* must have an area of 1 m^2, and at the same time the relation or ratio between the sides must be $b = \sqrt{2} \cdot a$

$$\left.\begin{aligned} a \cdot b &= 1\ m^2 \\ b &= \sqrt{2} \cdot a \end{aligned}\right\} \Rightarrow a \cdot a \cdot \sqrt{2} = 1\ m^2 \Rightarrow a^2 \cdot \sqrt{2} = 1\ m^2 \Rightarrow a^2 = \frac{1\ m^2}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow a = \sqrt{\frac{1\ m^2}{\sqrt{2}}} \Rightarrow a = \frac{1\ m}{\sqrt[4]{2}} \Rightarrow a = \frac{1}{1.189}\ m \Rightarrow a = 0.841\ m.$$

By knowing the value of *a* we can easily calculate that of *b*:

$$\left.\begin{aligned} a \cdot b &= 1\ m^2 \\ a &= 0.841\ m \end{aligned}\right\} \Rightarrow b = \frac{1\ m^2}{0.841\ m} \Rightarrow b = 1.189\ m.$$

Therefore the DIN A0 format has the following measurements:

$$\text{DIN A0} \begin{cases} \text{width} = \dfrac{1}{\sqrt[4]{2}}\ m = 0.841\ m \\ \text{length} = \sqrt[4]{2}\ m = 1.189\ m. \end{cases}$$

Direct and indirect measurements

A measurement can be direct, such as the measurement of temperature by a thermometer, or indirect, that is, if other measurements are needed in order to obtain the one required. If the measurement is taken by means of an instrument created specifically for that purpose then we refer to it as *direct measurement*. In this case we have an instrument which takes the measurement by comparing the quantity (the variable) which is to be measured with another of the same physical nature. This is what happens, for example, if the length of an object is compared with the length of a marked (calibrated) standard by making a comparison of distances.

Measurement techniques are strategies used to determine measurements, such as counting, estimating, using formulae and making use of instruments. These instruments are what most people associate with measuring. We are all familiar with rulers, tape measures, measuring spoons, scales, thermometers, stop watches and so on.

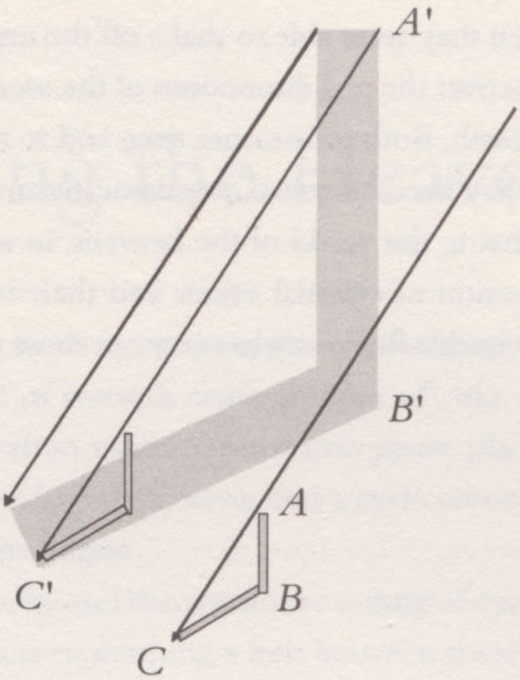
It may be that we are not able to take a direct measurement, either because there are magnitudes (variables) that we cannot measure by direct comparison with standards of the same nature, or because we are faced with a situation in which the quantity that we wish to measure is too big or too little, and there is no adequate tool available. In such situations the measurement will have to be made by means of a variable that enables another variable to be calculated. In this case we will be carrying out an *indirect measurement*.

The triangle plays an important role in the use of formulae and ratios for indirectly calculating measurements. The history of mathematics of full of examples. One instance is the Pythagorean theorem, which was subject to numerous proofs carried out by different cultures spread throughout time and geography. Mathematicians in ancient Egypt, Greece, Africa, China, India and Europe were all aware of it. The theory of similar triangles, or Thales' theorem, is another example in which the triangle features and which also provides techniques for taking indirect measurements.

The triangle was also fundamental in the development of the history of trigonometry. Linked for many centuries to astronomy, trigonometry provided the foundations for making calculations in measuring the heavens. It also forms the mathematical foundations of *triangulation*, the method used to measure the terrestrial meridian arcs which we shall deal with in the coming chapters.

Let's look at, for example, a calculation of indirect measure using the mathematical concept of similarity between figures, in this case right-angled triangles. We want to measure the height of a very tall tower (or building). For some reason or other it turns out that we cannot go up the tower to take a direct measurement – such as by hanging a cord or tape measure from the top of it. Even so, we can use a simple indirect method to find out the height.

Near the tower we place a vertical object (a rod, or a stake, for instance) and we measure its height. If we then measure the length of that object's shadow and the length of the building's shadow, we can discover the height we are looking for. How? Given the enormous distance separating the Sun from the Earth (150,000,000 km, give or take) we can consider the Sun's rays that fall on the tower and those falling on the object as being parallel to each other. The ratio between the height and the shadow of the object will be the same as the ratio between the height and the shadow of the tower or building, as the two triangles that we are comparing are similar (both are right-angled and the other two angles are equal). Therefore, it is enough just to calculate a ratio.



Measuring the tower by comparing its shadow with that of a gnomon, or shadow staff.

If $A'B'$ is the height we want to find, $B'C'$ is the length of the shadow of the tower (or building), AB is the height of the vertical object (rod, or gnomom) and BC is the length of its shadow, then as

$$\frac{A'B'}{B'C'} = \frac{AB}{BC}$$

so we can say

$$A'B' = \frac{AB}{BC} \cdot B'C'.$$

This simple mathematical reasoning enables us to calculate the height of the tower (or building) by means of an indirect measurement by using other direct measurements of lengths which we do have the possibility of measuring.

As we have just seen, mathematics makes an ideal tool for determining exact measurements. Throughout their history, human beings have attempted to find an explanation for the world, to understand their environs and control them. One of the most ancestral and universal preoccupations was to be able to measure time and establish a calendar. Another constant concern was to find explanations for the world in which they lived and to enjoy a global view of that world, in other

words, cosmology. When they were able to shake off the mythological explanations, humankind tried to discover the real dimensions of the world they inhabited and to measure the shape of Earth. Both to measure time and to measure the space of the physical world where they lived, they had, paradoxically, to look at a world that was inaccessible to them, that is, the world of the heavens. In some way, as we shall see later, systematic observation of celestial events and their interest in understanding them would eventually enable humanity to carry out these measurements and more.

Chapter 2

Measuring the Heavens

Civilisations in ancient times were already aware that many natural phenomena are cyclical. After a period of warmth came another of cold, and then warmth again followed by cold. Vegetation seems to burst into green, plants flourish, fruits appear and then, gradually, trees lose their leaves and a cycle comes to an end, only later to be repeated over and over again.

Observations of the sky and the systematic noting of cyclical changes that can be seen there led to humans establishing a link between those events and the changes they saw in the environment in which they were living. Recognition of this led humankind to study the position and motion of the heavenly bodies (astronomy) and enabled them to predict natural phenomena taking place on Earth, such as the changes of the seasons. Thus the idea took hold that heavenly bodies influenced terrestrial events (astrology).

One way or another, it became essential to measure the heavens. Mathematics and, in particular, geometry and trigonometry, turned out to be the most useful tools for this, and the Greeks came to develop a sophisticated mathematical astronomy which served them as an explanation for the apparent movements of the stars and, in particular, the motion of the planets.

Greek rational thought and cosmology

If we understand science as understanding, description and systematic explanation of natural phenomena, with the aid of logic and mathematics, then the origins of western science must be traced to the Greek and Hellenistic traditions. The roots of modern physics are embedded in the study of astronomical questions raised by the motions of the heavenly bodies and attempts to explain them rationally by means of mathematical models.

In general terms, the ancient conceptions of the universe (cosmologies) were markedly of a mythological nature, as humankind chose explanations involving the supernatural. Along with the philosophers of ancient Greece, however, was to come a rational cosmology. From the 6th century BC, the imaginative minds

of the great Greek thinkers attempted to provide a rational explanation for the phenomena without resorting to mythology. Natural phenomena were believed to be governed by determinable cause-effect relationships. They related changes in nature to underlying principles that they believed to be intelligible and which explained why events happened as they did.

ANAXIMANDER AND ANALOGICAL REASONING

The Ionian philosopher Anaximander (c. 610 BC–c. 545 BC) made use of reasoning by analogy. He was using this type of reasoning when he declared that "stars are portions of compressed air, in the shape of wheels, full of fire, which at times emit flames from small openings". Such an explanation now might make us smile but, situated in the context of his era, it signified an important step forward by ruling out supernatural forces and replacing them with natural causes.

Detail of The School of Athens *(1510–1511) showing Anaximander, by Raphael.*

The two great features of the heavens

To get an idea of the capacity of the rational Greek mind, which imagined sophisticated mathematical models for explaining, quantifying and predicting celestial phenomena, we should, just for a moment, shake off the knowledge we have and try to place ourselves in the context of the beginning of the 4th century BC. Only by doing this can we really understand the ingenuity of their achievements. By about 400 BC, the Greeks already had enough data on the apparent motion of the heavenly bodies, and it was then that they started to formulate mathematical theories on them. What were those data? In the parlance of what were the 'appearances' that 'had to be saved' – in other words, what was the rational explanation for observations?

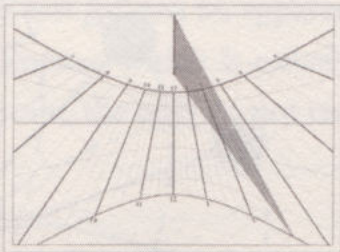
From systematic observation of the sky we can distinguish two important facts. The first of them is related to the diurnal motion of the Sun and the motion of

the stars; the second one concerns the motion of the planets. Let's look at the main observation data on both of these that were available to the astronomers of antiquity in Egypt, Mesopotamia and Greece.

The Sun's diurnal motion and the motion of the stars

Systematic local observation of the daytime motion of the Sun with the aid of a gnomon (a vertical rod on a horizontal surface) shows that the length and direction of the gnomon's shadow simultaneously vary – slowly but continuously – throughout the day (the period of daylight) and thus determine the direction of the Sun.

The movement of the shadow of the gnomon during the day follows a symmetrical figure in the shape of a fan. That figure changes every day, but at the time of day when shadow of the gnomon is shortest, the shadow is always pointing in the same direction.



On the left: A monumental sundial at the Chinook Trails Elementary School in Colorado Springs (United States). Above: Lines for the projections from the tips of the shadow of the gnomon.

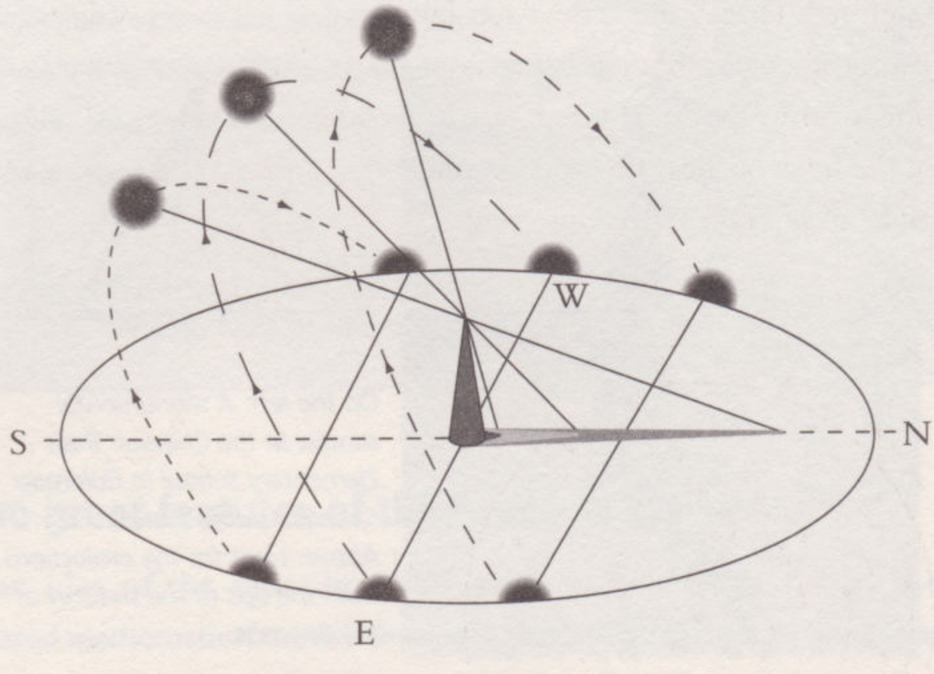
This enables us to identify the *direction* of north (shown daily by the gnomon's shortest shadow) and, from that, the directions south, east and west; the *noon* of the location (the instant at which the gnomon's shadow is shortest), and the *solar day* (the interval of time separating, at a determined point, two consecutive noons, and which is of 24 hours).

At the same time, the position of the rising Sun on the horizon varies from one day to the next: it moves gradually and cyclically from the cardinal point east (*spring equinox*) towards a point situated further north (*summer solstice*), from where

it goes to the east (*autumn equinox*) and continues towards the south, to a point from which it again changes direction (*winter solstice*), then to go again towards the east, thus repeating the cycle. The position of the setting Sun varies analogously around the west point. A year could thus be defined as the period of time separating two consecutive spring equinoxes.

The daily hours of sunlight vary from one day to the next. The winter solstice is the day on which the period of sunlight is the shortest in the year, and the gnomon's shadow at noon is the longest in the year. The summer solstice is the day on which the period of sunlight is the longest in the year, and the gnomon's shadow at noon is the shortest in the year.

In summary, the variations in the positions of the rising (and setting) Sun on the horizon correspond to the seasonal cycle, and the Sun crosses the sky once every 24 hours, changing its elevation with the seasons.



The apparent motion of the Sun.

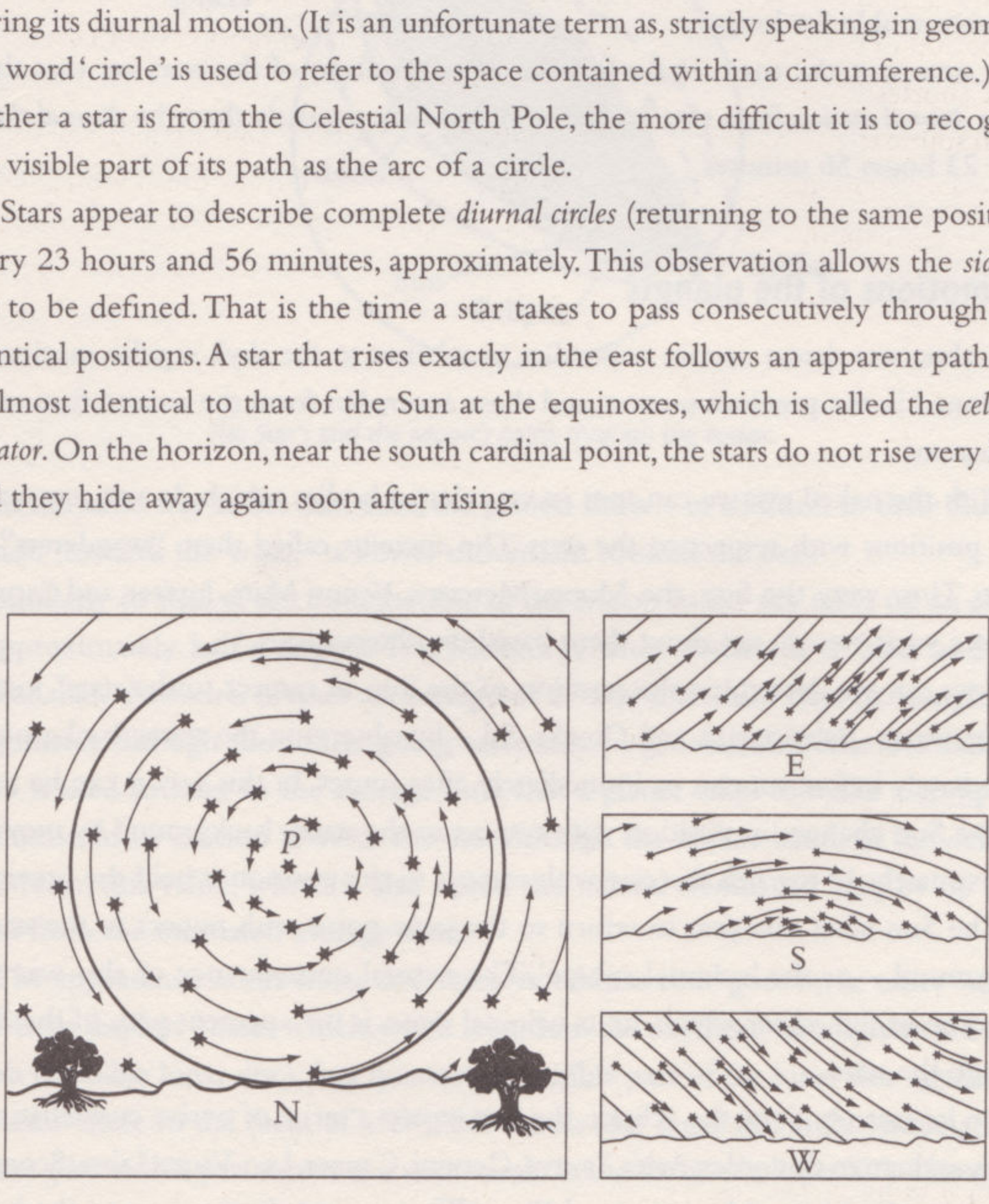
Systematic observation of the night sky indicated that the relative positions of the stars are invariable. This enabled observers to refer to constellations, clusters (arbitrary groups) of neighbouring stars, and to set out a stellar map.

The stars move globally from east to west. This movement is called *diurnal motion* (or daily motion) because it resembles the diurnal motion of the Sun, which is also westwards.

Far above the horizon, to the north, there is a point *P*, very near to Polaris, which we shall call the Celestial North Pole, around which the nearby stars appear to rotate in circular arcs. The stars whose angular distance in respect to the Celestial North Pole is less than or equal to the distance between it and the horizon (*N*) never disappear from sight below the horizon, as can be seen in the following figure. They are, therefore, stars that are visible at any hour on any night (i.e. on a night with good visibility and a clear horizon). Such stars are known as *circumpolar stars*.

The name *diurnal circle* is given to the circle that is seemingly followed by a star during its diurnal motion. (It is an unfortunate term as, strictly speaking, in geometry the word 'circle' is used to refer to the space contained within a circumference.) The further a star is from the Celestial North Pole, the more difficult it is to recognise the visible part of its path as the arc of a circle.

Stars appear to describe complete *diurnal circles* (returning to the same position) every 23 hours and 56 minutes, approximately. This observation allows the *sidereal day* to be defined. That is the time a star takes to pass consecutively through two identical positions. A star that rises exactly in the east follows an apparent path that is almost identical to that of the Sun at the equinoxes, which is called the *celestial equator*. On the horizon, near the south cardinal point, the stars do not rise very high and they hide away again soon after rising.



Apparent motion of the stars depending on the viewing direction: north (left), east (top right), south (centre right) and west (bottom right).

The Greeks knew that if they moved from where they had been making their observations and went in a southerly direction, towards Egypt, for example, the height of the Celestial North Pole decreased at a rate of 1° approximately every 110 km. Certain stars that before were circumpolar ceased to be so; stars which before rose at the east cardinal point and set in the west continued to do so, but followed a path that became more and more perpendicular to the plane of the horizon. Finally, they saw more and more stars which before had not been visible at all, and those stars near to the south cardinal point rose to a greater elevation and were visible for longer.

In summary, the main characteristics of the motion of the stars are that they have a *diurnal motion* from east to west, and that they regularly describe *diurnal circles* every 23 hours 56 minutes.

The motions of the planets

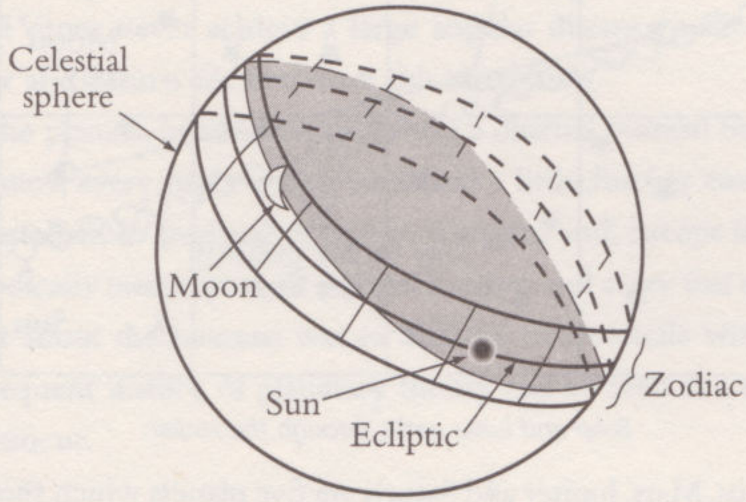
Keen observers detect certain anomalies in relation to the sky's regular motion as mentioned in the previous section, and these anomalies form the second feature of the heavens.

With the naked eye we can spot seven celestial bodies which do not remain in fixed positions with respect to the stars. The ancients called them "wanderers" or *planets*. They were the Sun, the Moon, Mercury, Venus, Mars, Jupiter and Saturn. Let's see what we can say about them based on observations.

How can we determine the position of the Sun in respect to the stars? Just as the Egyptians, Babylonians and Greeks did – by observing the starry background immediately before sunrise, or immediately after sunset. In this way it can be seen that the Sun changes its position with respect to the starry background by moving approximately 1° towards the east with respect to the position it held the previous day. The Sun takes one year to return to the same point with respect to the starry background – or the 'celestial sphere'. The natural consequence of this was the definition of the *ecliptic* which, in its original sense, is the apparent path of the Sun through the stars.

On its journey along the ecliptic, the Sun crosses a series of twelve constellations. It crosses them in this order: Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius and Pisces. The name *zodiac* is given to the band some 16° wide containing those twelve constellations, all of them familiar from popular astrology.

In its apparent motion along the ecliptic, the Sun arrives at the celestial equator at the equinoxes and moves away from it up to a maximum of around 23.5° at the two solstices (although in opposite directions for each one). The Greeks found that the Sun's apparent speed along the ecliptic was slightly higher in the Greek (northern) winter than in summer.



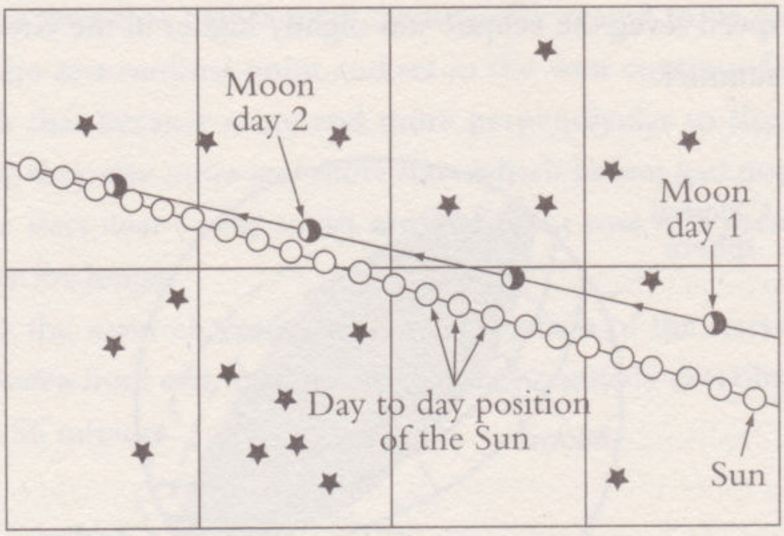
The Sun's and the Moon's paths through the zodiac.

In the same way as the Sun does, the planets show – in addition to their diurnal motion (towards the west) – a slower movement towards the east.

Similarly to that of the Sun, the disc of the Moon in the sky takes up an angle of approximately half a degree. The Moon's journey eastwards is faster and less regular than the Sun's. It takes an average of 27 and one-third days to complete a full journey through the zodiac going eastwards. In general, the term *mean sidereal period* is used to refer to the average time that a planet takes to make a complete revolution in its motion towards the east through the zodiac band. In the case of the Moon, this value, which is also called the *sidereal month*, can vary by up to 7 hours from the estimated average time.

The appearance of the lunar disc varies notably as time goes by, in other words, the Moon displays phases which, when seen from the northern hemisphere are new moon (invisible lunar disc), first quarter (the visible part of the lunar disc appearing as a semi-circle in the form of a 'D'); full moon (the lunar disc fully visible); third quarter (the visible part shows as a semi-circle in the form of a letter 'C'). The cycle between two consecutive full moons (*lunation* or the Moon's *mean synodic period*) lasts an average of 29 days and a half. The real time can vary up to half a day from that average. Finally, it can be seen that the Moon's path in respect to the starry

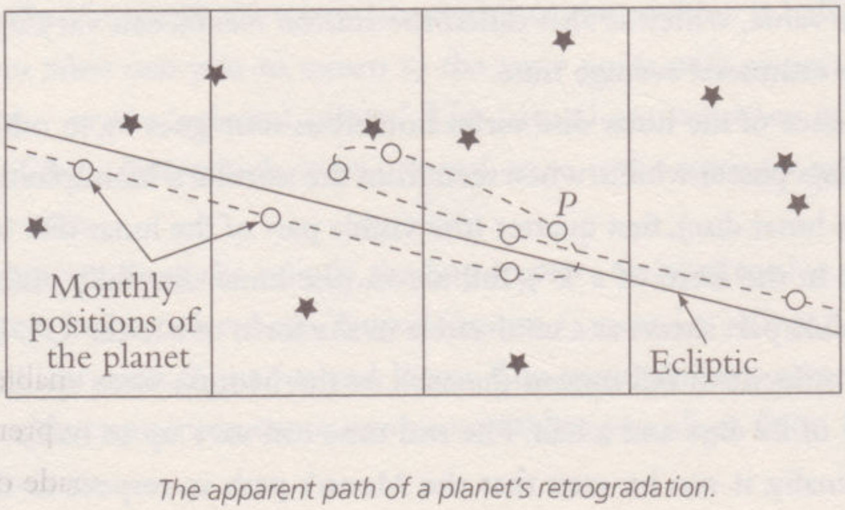
background at times coincides with the ecliptic and then moves away from it to a maximum value of approximately 5°, first in one direction and then the other.



Solar and lunar paths through the zodiac.

Mercury, Venus, Mars, Jupiter and Saturn are five planets which show up as simple bright points in the sky. Calculation of their mean sidereal periods produces a range of different values: for Mercury it is 1 year; Venus 1 year; Mars, 687 days; Jupiter, 12 years, and for Saturn 29 and a half years, though in all cases their real sidereal periods can vary from those given here.

Planets' eastwards motion is called *normal motion* or *proper motion*. It can be observed that none of these five small planets has a constant speed in its proper motion. Furthermore, and this was the most surprising observation of all, they display retrogradations, periodic interruptions in their proper motion towards the east when for certain periods of time they moved towards the west by making a



The apparent path of a planet's retrogradation.

kind of loop in their apparent path before recovering their 'proper' motion. During these retrogradations, the planets shine more brightly.

Mercury reverses the direction of its normal motion every 116 days; Venus does it every 584 days; Mars every 780 days; Jupiter, every 399 days; and Saturn every 378 days. These values are those of their *mean synodic periods*, in other words, the average time passing between two consecutive retrogradations.

Mercury and Venus never achieve a large angular distance away from the Sun, but Mars, Jupiter and Saturn are free from this restriction.

To sum up, the planets, in addition to having a diurnal motion towards the west along with the stars, every night are to be found a little further east in relation to the zodiacal constellations (normal or proper motion) and, except for the Sun and the Moon, periodically interrupt their normal motion and carry out retrogradations. This second fact about the heavens was so difficult to reconcile with the first one that all the subsequent history of planetary theory can be seen as a series of efforts to make them concur.

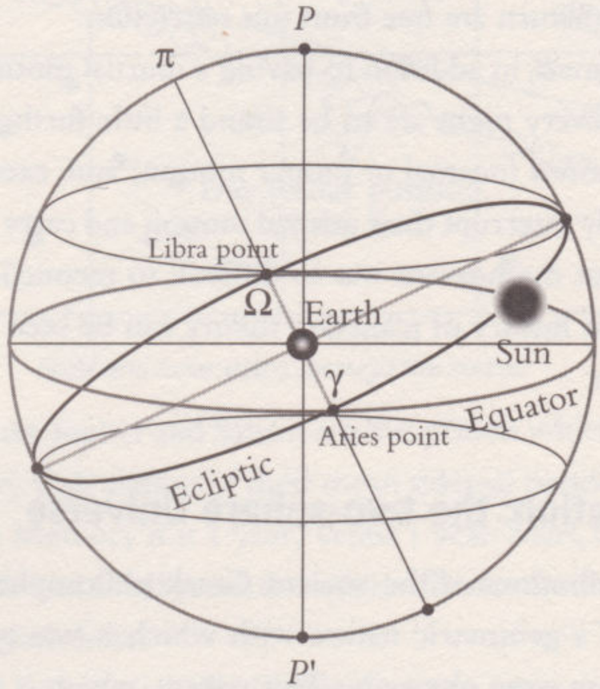
The first explanation: the two-sphere Universe

Over time, the contributions of the ancient Greek philosophers began to form a conceptual model of a geometric nature with which it was possible to explain a large part of what they were observing. This system, which is called the *two-sphere universe*, was accepted from the 4th century BC onwards by the majority of Greek philosophers and astronomers. It consists of considering the Earth as a motionless sphere situated in the centre of a much larger sphere (the celestial sphere) which rotates westwards around a fixed axis – which passes through the Celestial North Pole – and drags the stars with it. Beyond the celestial sphere there is nothing, neither space nor matter.

Within this structural framework, different and contradictory astronomical and cosmological systems were to appear during the almost two thousand years separating the 4th century BC from the time of Nicolaus Copernicus (1473-1543). But once the actual framework itself had been established, its veracity was hardly ever questioned.

This model does not provide an explanation for the motion of all the heavenly bodies, particularly the motions of the planets but, for one thing, it gives an astonishingly simple explanation for the first feature of the heavens. It enables a huge amount of individual observations to be ignored as long as just a few premises are taken into account and, on top of that, it enables predictions to be made on future

positions of the stars. The few premises we need are that the celestial sphere makes a complete revolution, towards the west, every 23 hours and 56 minutes, and that the Sun follows, in a year and eastwards, the perimeter of a great circle (ecliptic) inclined some 23.5° (in actual fact, 23° 27'), with respect to the equator of the sphere (the celestial equator). Once the position of the Sun is fixed on the ecliptic, that day it describes a circle running parallel to the equator.



The two-sphere universe.

The geometric model of the two-sphere universe is not something just belonging to the past. It continues to be used in modern observational astronomy on account of its simplicity and usefulness in determining the position of the heavenly bodies. Measurements made to determine the stellar positions (the coordinates) are angular, so we can imagine them as being on a sphere.

The Greeks put forward powerful reasons in defence of this idea of the universe of two spheres. Aesthetic arguments were of great value in Greek culture, and it should come as no surprise that they are used here, too. The shape of a sphere is used both for situating the stars and for the shape attributed to the Earth. A sphere was considered by Greek geometrists to be the most perfect figure, the one that always occupies the same space when rotated on its axis. More than that, a celestial sphere made sense because it could be seen that in their apparent motion the stars seemingly moved in circles. The Earth had to be spherical, not only for aesthetic

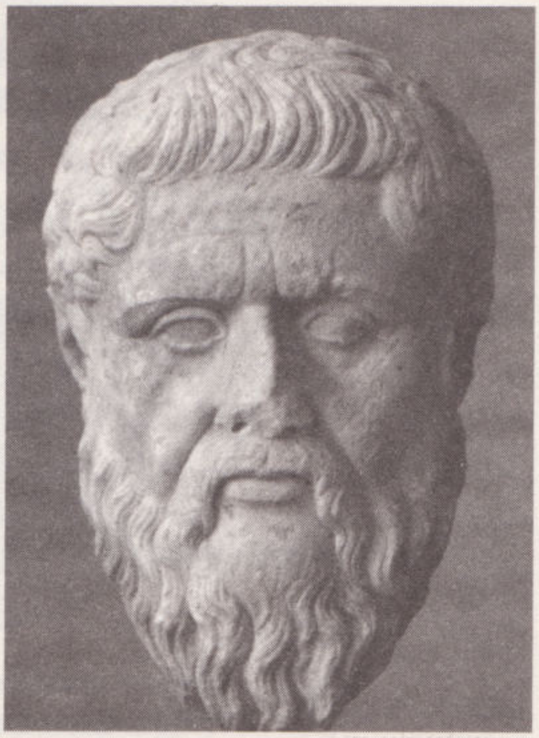
reasons but because, when viewed from a high place on land, it could be seen how the hull of a departing ship disappeared before the mast did, and that the mast appeared first when a ship was approaching. Furthermore, the shadow that the Earth cast onto the Moon during lunar eclipses always had a curved edge.

Surely the Earth had to be in the centre (geocentrism) not only to maintain the model's symmetry but also because a body situated in the centre of a sphere has no direction in which to fall? All directions point upwards, therefore, the Earth cannot fall in any direction and must remain in the centre. Lastly, the absence of motion of the Earth (geostaticism) was explained on the basis of observations made on the positions of the stars and issues of common sense concerning objects situated in the air, or the trajectories followed by stones tossed straight upwards.

As for the stars, the data that were available did not enable any changes in relative distances between stars to be detected (the absence of stellar parallax), which would be what would be seen if the Earth was moving. We know today that the changes could not be detected on account of the huge distances from the Earth to the stars. Finally, if the Earth did move, bodies situated in the air, such as birds or stones tossed upwards, should be left behind, due to the Earth's motion and, if the Earth were rotating, objects that were not tied down would be flung off. We would also be feeling a perpetual strong wind that the Earth's motion would necessarily stir up. However, we see none of that happening; therefore the Earth does not move.

The second explanation: geometric astronomy

Plato (427–347 BC) set the foundations for what we could call a research programme in Greek astronomy when he – supposedly – put this question to scholars: "What are the regular and orderly movements the supposition of which makes it possible to *save the phenomena* (rationally explain the apparent facts) concerning the movements of the planets?"



A Roman copy of the head of Plato that was displayed at the Athens Academy after his death.

The principle of circularity

For Plato, the founder of the *Academia* in Athens, the truth was to be found in the world of ideas, of pure forms, and he was disdainful of experimentation. We can state three characteristics of the Platonic approach which influenced, to a greater or lesser degree, subsequent astronomy and cosmology. The first was scorn or distrust of observation; the second was the conviction that the cosmos was structured in accordance with a perfect geometry; and the third was the establishment of the basic principle of circularity and uniformity – all the celestial bodies move in a circular and uniform motion. The Platonic cosmological vision can be seen from the ideas that he set down in some of his dialogues, in particular the *Phaedrus*, the *Phaedo*, *The Republic* and the *Timaeus*.

In *The Republic*, Plato speaks of a spindle in which another spindle fits, and so on up to eight, with their rims showing as circles. Later, he writes: "The whole spindle revolved with a single motion, and within it, the seven inner circles also revolved slowly, but in the opposite direction to the whole." It is clear that the inner spindles he is referring to are the planets. From Plato onwards, any serious astronomy or cosmology would include the erratic motions of the planets. The subsequent influence of Plato's postulate relating to the circular and regular motion of the planets was extraordinary. The fallacy of circularity held the world of astronomy under its spell for two thousand years.

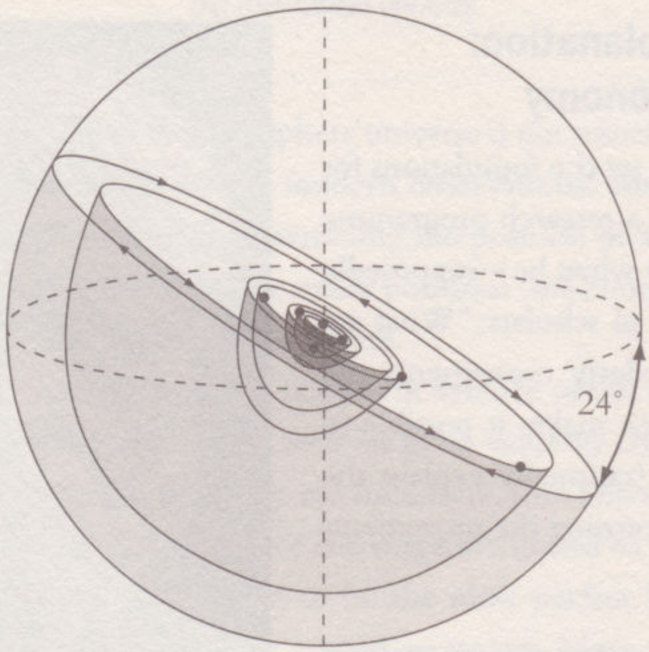


Diagram of the Platonic cosmology deduced from the ideas set out by Plato in his dialogues.

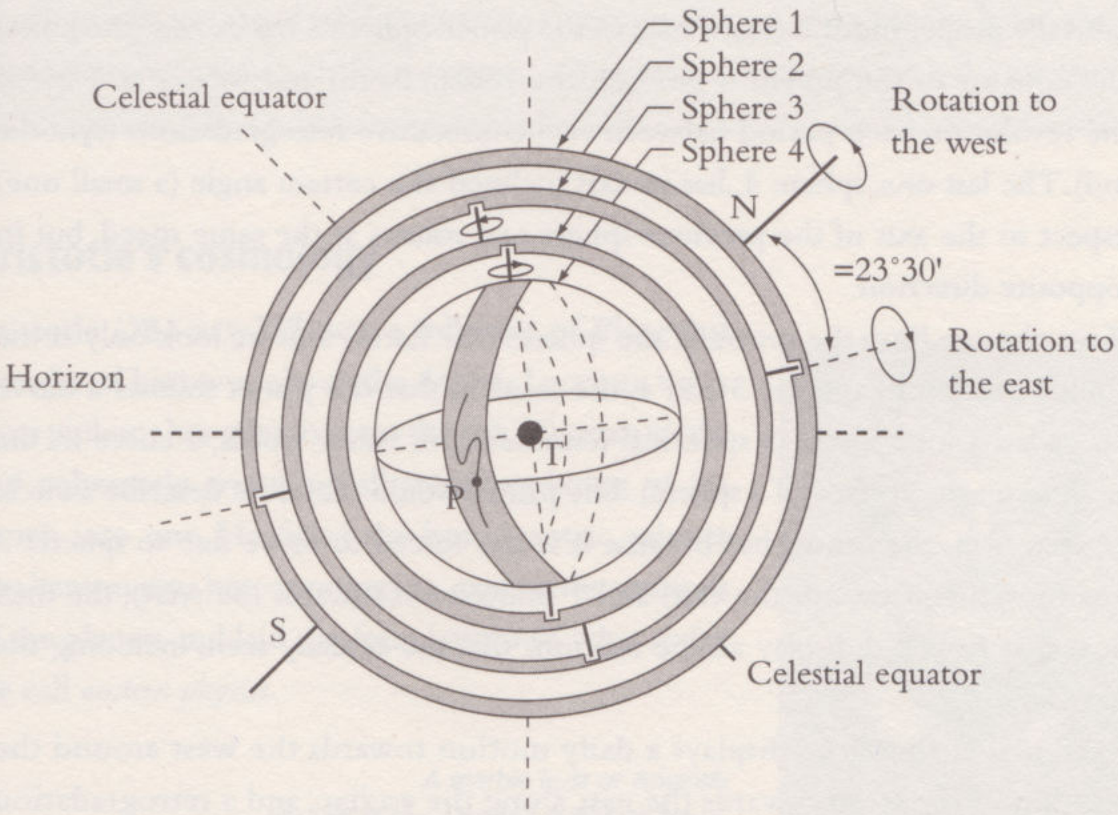
The homocentric spheres system

The mathematician Eudoxus of Cnidus (c.390–c.337 BC) was the first to provide a serious reply to the question supposedly posed by Plato. He proposed an ingenious system of concentric spheres with which he managed to provide a first explanation – and a very ingenious one – for the motion of the planets based on circular movements.

In his *system of homocentric spheres*, Eudoxus held that for every planet there was a model with a certain number of nested spheres, concentric to the Earth – three for the Sun, three for the Moon and four for each of the other planets (Mercury, Venus, Mars, Jupiter and Saturn). To explain the motion of all the stars, only one sphere was needed. Therefore, just to explain the celestial motions, he had to use a total of 27 spheres:

$$3 \text{ (Sun)} + 3 \text{ (Moon)} + 20 \text{ } (4 \times 5\text{, the five planets}) + 1 \text{ (the stars)} = 27.$$

However, Eudoxus did not link the movement of the set of spheres of one planet to another planet's set of spheres. They were mathematical models that were independent of each other.



The Eudoxian system of a planet's spheres.

In the cases of Mercury, Venus, Mars, Jupiter or Saturn, each of these planets was linked to four spheres in the following arrangement. The planet was situated at the equator of the innermost sphere (sphere 4); the axis of this sphere (the poles) was fixed to another sphere (sphere 3), which was concentric with it and bigger; the axis of sphere 3, in turn, had its poles joined to the next sphere (sphere 2), which was larger still and concentric with the previous ones; finally, and analogously, the axis of sphere 2 had its poles connected with the last sphere (sphere 1), which was larger than and concentric with all the others.

In this way, the axis of each sphere (and therefore, each pair of poles) is forced to move by the movement of the nearest sphere surrounding it. All the spheres rotate at constant yet respectively different speeds around their own axes.

What part is played by each of the four spheres in the explanation of the planet's motion?

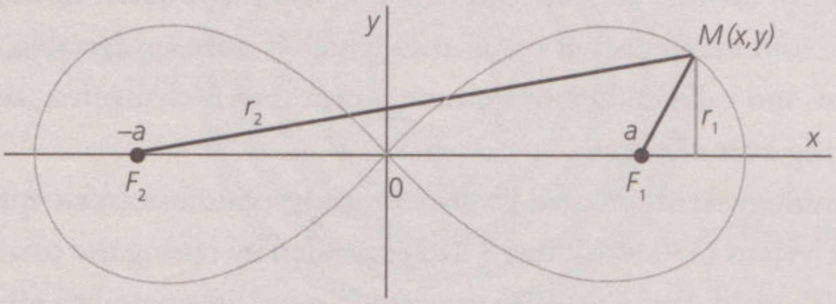
The first one, let's call it sphere 1, has its axis in the direction north-south and rotates daily from east to west; it explains the planet's diurnal motion, corresponds to the sphere of the stars, and is the one that moves the other spheres. Sphere 2 has its axis inclined with respect to the previous one at an angle that is almost equal to the angle formed by the ecliptic with the celestial equator, and it rotates towards the east at a speed of one revolution for each of the planet's sidereal periods. That explains the proper motion (eastwards) of the planet. Sphere 3 has its axis (the poles) on the equator of the previous one (on the zodiac band) and rotates at a speed of one revolution each period between two consecutive retrogradations (synodic period). The last one, sphere 4, has its axis inclined at a certain angle (a small one) in respect to the axis of the previous sphere and rotates at the same speed, but in the opposite direction.

If we observe from the centre of the spheres (the Earth) and we look only at the combined motion of spheres 3 and 4, the result is that the planet follows a curve that is called a hippopede (a spherical lemniscate, in other words, a curve in the shape of ∞ on the surface of a sphere). The planet would therefore describe a circle when seen from the centre, but because it is also forced to move due to spheres 2 (a slow movement towards the east) and 1 (movement towards the west), the final result is that it would display all the motions that are actually seen, including the retrogradations.

Each planet therefore displays a daily motion towards the west around the Earth, a proper motion towards the east along the zodiac, and a retrogradation motion.

THE LEMNISCATE

The hippopede or spherical lemniscate appears in Eudoxus's model of homocentric spheres. On the plane, a lemniscate is a curve with a characteristic shape consisting of two loops that meet at a central point, as shown in the figure.



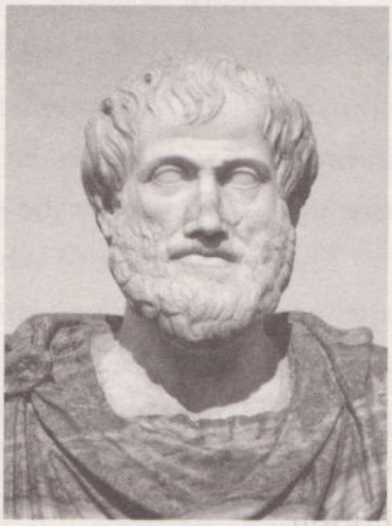
On the plane, a lemniscate can be described by this general equation:

$$(x^2+y^2)^2 = 2a^2(x^2-y^2),$$

with $2a$ being the distance between the foci F_1 and F_2. The first description of the curve, which is also known as the lemniscate of Bernoulli, was given in 1694 by the Swiss mathematician and scientist Jakob Bernoulli (1654–1705) who described it as the modification of an ellipse. If the ellipse is the curve that is defined as the series of points on the plane such that the sum of their distances to two fixed points F_1 and F_2 (foci) is a constant, the lemniscate is the series of points on the plane such that the product of their distances to the two foci is also constant.

Aristotle's cosmology

Aristotle (384 BC–322 BC), a follower of Plato but founder of his own school, the Athens Lyceum, was a great philosopher who took on the task of systematically and coherently arranging all the knowledge of his epoch into one. He fitted the homocentric spheres mechanism into his cosmology to explain the motions of the planets and laid the foundations of what today we call *ancient physics*.



A marble bust of Aristotle housed at the Altemps Palace in Rome.

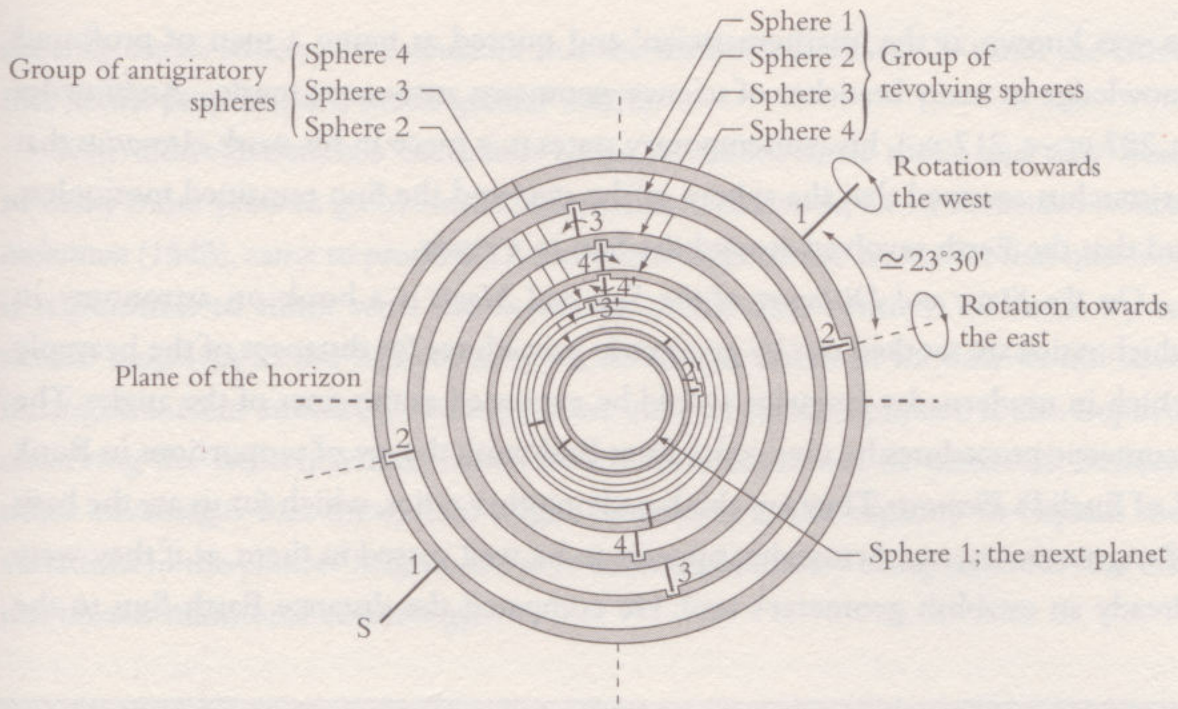
According to Aristotle, an essential difference exists between the sublunary region that stretches from the Earth to the Moon and the celestial (or 'superlunary') region. There is a clear distinction between the eternal, regular and changeless revolutions of the celestial spheres and the motions of the terrestrial region, which are finite, irregular and transitory. Objects in the sublunary region are made up of four elements: earth, water, air and fire. The celestial bodies, however, are not made of those four elements but of another which is pure, unalterable, transparent and weightless, and called ether, or quintessence. Ether is changeless, and so in the heavens nothing ever changes.

In his cosmology, Aristotle used Eudoxus's model of homocentric spheres which Callippus of Cyzicus (370 BC–300 BC) had expanded by raising the total number of spheres from 27 to 34. But Aristotle's systematic mind attempted to give a physical reality to the geometric devices that appeared in them. A geometric scheme could only be acceptable if it had a mechanical sense and fitted in with our general ideas on matter and motion.

He therefore constructed a single system of homocentric spheres which were all connected together and acted simultaneously. The celestial sphere, the outermost one, imposed a westward motion on all the rest. To prevent any sphere associated with a determined planet from imposing its particular motion on all the lower spheres, he introduced, between the series of spheres of each planet and that of the next planet (i.e. the inner one), compensating spheres which rotated on the same axes and at the same speeds as each of the planetary spheres of the previous (outer) set, but in the opposite direction. The introduction of this series of counter-rotating spheres brought the total number of spheres to 56, which were now in physical contact and formed a global system.

The system of homocentric spheres already had its critics back in ancient times because in this system the distance between a determined planet and the Earth is constant and it was therefore not easy to explain, among other things, the variation in brightness of the planets during the retrogradations, which was a phenomenon that they explained by the planets coming nearer to the Earth.

The foundations of what we call ancient physics were laid by Aristotle, who analysed and explained the motions by clearly differentiating between celestial dynamics (superlunary region) and terrestrial dynamics (sublunary region). His physics doctrine was accepted as dogma by 60 generations because it was a very elaborate and coherent theory, which coincided with common sense and everyday experience, and which he integrated into his cosmology.



Aristotle's counter-rotating spheres. If the first four spheres correspond to Saturn, then, except for sphere 1, which imposes the movement to the west on the others, the motion of the other three (2, 3, 4) is nullified for the planet that follows by three counter-rotating spheres (4', 3' and 2').

Aristotle not only accepted that the cosmos is geocentric, geostatic and fundamentally circular but also spoke out for these postulates with outstanding skill and ingenuity. Aristotelian cosmology established a connection between astronomy and physics that resulted in a coherent whole, a true world system, a cosmophysics. It is therefore no wonder that, apart from a few rare exceptions such as that of Aristarchus of Samos, practically all Greek, Arab and European astronomers accepted, implicitly or otherwise, the fundamental premises of Aristotelian cosmology: the closed and finite nature of the cosmos, the motionlessness of the Earth in the centre of the Universe, and the essential difference between two regions, the celestial (superlunar) and the earthly (sublunar).

Aristarchus of Samos

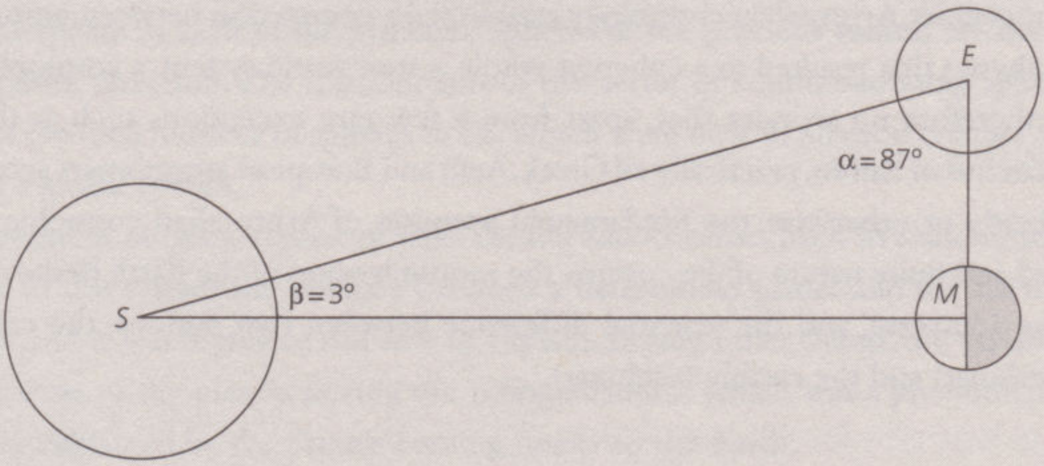
Very little is known about the life of Aristarchus of Samos (c.310 BC–c.230 BC), who was a disciple of Strato of Lampsacus, the third director of the Lyceum. What we know is based on his work *On the Sizes and Distances of the Sun and Moon* and quotations found in later works. Acknowledged within his epoch as an astronomer,

he was known as the 'mathematician' and quoted as being a man of profound knowledge in many branches of science: geometry, astronomy, music... Archimedes (c. 287 BC–c. 212 BC), his contemporary, states in a piece in his work *Arenarius* that Aristarchus assumed that the sphere of the stars and the Sun remained motionless, and that the Earth revolved around the Sun in a circle.

On the Sizes and Distances of the Sun and Moon is a book on astronomy in which ratios are worked out by geometric procedures for distances of the heavenly which in modern-day language would be expressed as the sines of the angles. The geometric procedures he uses come from Eudoxus's theory of proportions in Book V of Euclid's *Elements*. They are also based on other ratios, which for us are the basis of trigonometry, and Aristarchus appears to be well versed in them, as if they were already an establish geometer's tool. He compared the distance Earth-Sun to the

MEASUREMENT OF ARISTARCHUS'S EARTH–MOON AND EARTH–SUN RELATIVE DISTANCES

In the 3rd century BC, Aristarchus of Samos calculated how much further away the Earth is from the Sun than from the Moon, and also their relative sizes. For this task he took into account that when the Moon is 'half-full' (the crescent moon) the triangle *EMS* is right-angled (the angle Earth–Moon–Sun is 90°). He measured the angle α then formed by the lines of vision to the Sun and to the Moon, which he calculated at 87°. As the sum of the angles of a triangle is 180°, $\beta = 3°$.



In this way he was able to calculate the ratio of distances $d(E,S) / d(E,M)$ through impeccable mathematical reasoning. As a simplification, and in present-day notation, the basic idea is:

$$\cos 87° = \frac{d(E,M)}{d(E,S)}.$$

distance Earth–Moon and calculated that the former was nearly 20 times the latter (the actual proportion is much greater 390:1).

Why didn't Aristarchus' successors adopt the heliocentric thesis, and why were so many more years to go by until Copernicus, in his work, *De revolutionibus orbium coelestium* (1543), came to propose a heliocentric hypothesis? To answer that question it is pointless to think with the mentality of the 21st century. We must try to situate ourselves in the 3rd century BC. To set the Earth in motion would have infringed ancient authority, common sense and Aristotelian physics. It also implied observing the stellar parallax, which at that time could not be observed. Besides, other advantages that the system might have, such as the capacity to explain the variations in the planets' brightness, were soon explained by using methods that did not breach traditional cosmology.

with $d(E,S)$ being the distance from the Earth to the Sun, and $d(E,M)$ the distance from the Earth to the Moon:

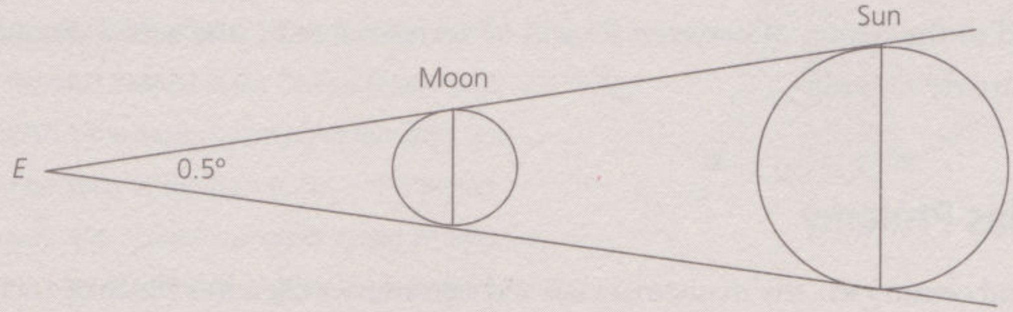
$$d(E,S) = \frac{1}{\cos 87^{\circ}} d(E,M),$$

and as $\frac{1}{\cos 87^{\circ}}$ is approximately 19, the result is that:

$$d(E,S) \approx 19\, d(E,M).$$

Additionally, as the Moon and the Sun, as observed from the Earth, are seen under the same angle of half a degree, the result is that the diameters of those heavenly bodies will have the same proportion:

diameter of the Sun = 19 diameters of the Moon.



The mathematical method was ingenious and rigorous, despite the fact that Aristarchus made a mistake when measuring the angle α, which is not 87° but 89° 52′ (the Sun is some 390 times further from the Earth than the Moon is).

Hipparchus of Nicaea

Hipparchus of Nicaea (c.190 BC–c.120 BC) used new geometric devices in astronomy and provided a quantitative evaluation of the irregularities in the motions of the Sun and the Moon. This was the prototype of Alexandrian astronomy which is characterised by its attempts to fully reconcile the principle of circularity with the events and facts observed. Keeping to that premise, under his astronomy programme, the astronomer has to define the number, size and positions of the circles under consideration, as well as the speed of circular motion that each one has, with the purpose of proving, by means of geometry and numerical calculation, that the system proposed is able to explain the appearances, make exact quantitative predictions and is capable of making predictive tables.

Hipparchus carried out outstanding astronomical observations, made improvements to the stellar map, collected a great number of observational data made by the Babylonians and discovered the precession of the equinoxes (the retrograde motion of the equinoxial points of intersection of the celestial equator with the ecliptic, causing earlier occurrence of the equinoxes, or the principle of the seasons).

By Hipparchus's time, calculations had already been made of the terrestrial circumference (Eratosthenes had done it, as we shall see in Chapter 4, which is entirely dedicated to measurements of the Earth). This enabled Hipparchus to calculate absolute values for distances to the Sun and to the Moon. By following his own methods and methods similar to those used by Aristarchus, he determined the size ratios for the Earth and the Moon. He observed the Earth's shadow on the Moon's silhouette at different phases of a lunar eclipse and, by taking into account that the Sun is very distant from the Earth and from the Moon, calculated that the terrestrial diameter was 8/3 that of the Moon (and not twice as big, as Aristarchus had estimated). For the distance to the Sun he obtained a value of 490 terrestrial radii, and to the Moon of between 59 and 67 terrestrial radii (the actual distance is approximately 60 radii).

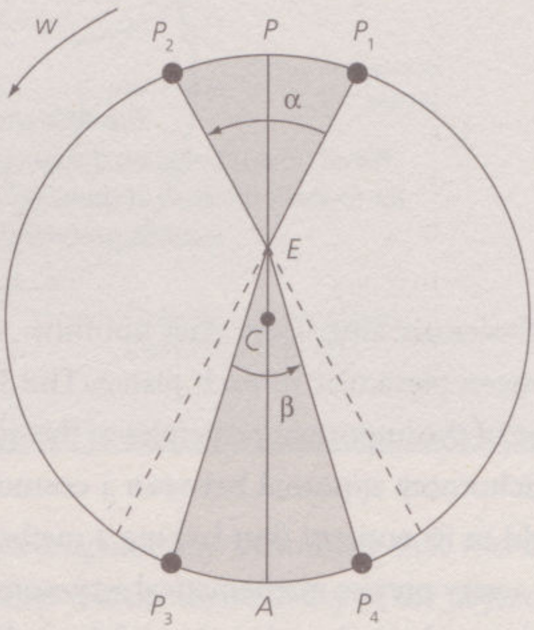
Claudius Ptolemy

In the 2nd century AD, the mathematician and astronomer Claudius Ptolemy (c. 100–c. 170) worked at the Museum and Library of Alexandria. He was responsible for the introduction of a way of working in practical astronomy that would continue right up to the 16th century. His greatest work, *Syntaxis mathematica* or *Almagest*, was the first systematic mathematical treatise to give a complete, detailed and quantitative

explanation of all the celestial motions. Ptolemy made certain principles of physics acceptable within astronomical hypotheses. It was not a case of just the principles of circularity and regularity but also other principles related to Aristotelian physics, such as geocentrism, the movement of stars fixed on a particular sphere, or the non-existence of the vacuum. In his planetary theory, Ptolemy used geometric devices that submitted questions of the real physical paths of the planets and the principles of Aristotelian physics to the precision of calculation. The models that he constructed made exact quantitative forecasts of future planetary positions.

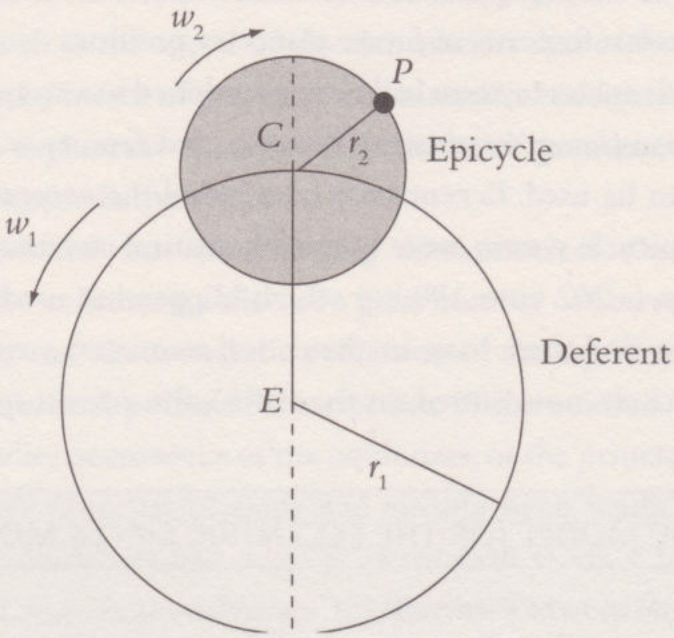
The homocentric spheres system had been abandoned in astronomy (it could not explain the changes in planets' brightness). From the 3rd century BC other geometric procedures began to be used. Eccentricity (also called the eccentric circle model) and the deferent-epicycle system, were two mathematical inventions introduced by Apollonius of Perga (c.262 BC–c.190 BC) which Hipparchus used in astronomy. In the *Almagest* we can find three basic mathematical resources: *eccentricity* (the planet is on an eccentric circle, not centred on the Earth), the *deferent-epicycle* system (the

THE ECCENTRIC MODEL (OR THE ECCENTRIC CIRCLE MODEL)



If the Earth (E) is taken to be motionless and a planet (P) is situated in an eccentric circular orbit, in other words, with centre (C) not coinciding with the Earth, an explanation can be found for the fact that the planets cover equal arcs in unequal times. That is so because, measured from the Earth, the apparent angular speed of a planet situated on an eccentric orbit is greatest at the point of the orbit nearest to the Earth (perigee) and is least at the most distant point (apogee), as can be seen in the figure. So, if the planet moves with constant angular speed w with respect to C, the time it takes to go from P_1 to P_2 is equal to the time it takes to go from P_3 to P_4, but both arcs P_1P_2 and P_3P_4 are not seen under the same angle from E. This technique allowed Hipparchus to explain the apparent motion of the Sun through the ecliptic, whose speed is not constant throughout the year.

planet is situated on a circumference, the epicycle, whose centre moves in turn in another circle, the deferent, in principle centred on the Earth), and the *equant point* (a point within the deferent and different from its centre, in respect to which the centre of the epicycle describes equal angles in equal times). Using these systems Ptolemy was not only able to explain all the 'appearances', but also to predict future planetary positions.



The deferent-epicycle set.
Planet (P) *is situated on the epicycle and revolves towards the west (or towards the east) at speed* w_2, *while the centre of the epicycle* (C) *revolves towards the east at speed* w_1.

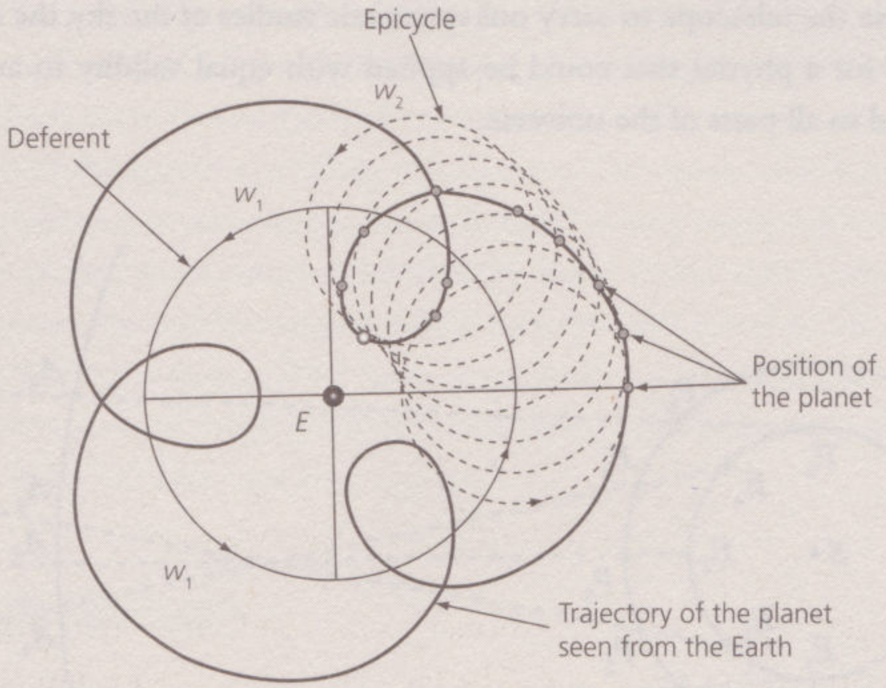
Ptolemaic astronomy does not form, in fact, an authentic system, but a series of solutions particular to each planet. The Ptolemaic system came into conflict with some of the important principles of the only known physics system, that of Aristotle. A dichotomy appeared between a cosmology that offered a physical system of the world in its entirety (but lacking a mathematical description of the observed facts) and a very precise mathematical astronomy able to account for the appearances (but lacking a physical explanation of the reality and the causes of the motions).

The Copernican system

The objections raised against the heliocentric hypothesis proposed by Aristarchus of Samos in the 3rd century BC were the same ones that from the times of Aristotle and

THE DEFERENT–EPICYCLE SYSTEM. EXPLANATION OF RETROGRADATIONS

The deferent-epicycle system provides an explanation for retrogradations and the variation in luminosity of the planets (interpreted as a variation in the proximity of the planet to the Earth). If we look at an ideal case in which the angular speed of the centre of the epicycle *C* in respect to the Earth is triple the angular speed of the planet in respect to *C* ($w_2 = 3w_1$), the result, seen from the Earth, would be that shown in the figure, and the planet would describe three loops coming nearer to the Earth. When we observe it against the starry background, the planet retrogrades and then shines more brightly as it is 'nearer' to the Earth. This simplified situation is quite similar to the model that would correspond to the planet Mercury.

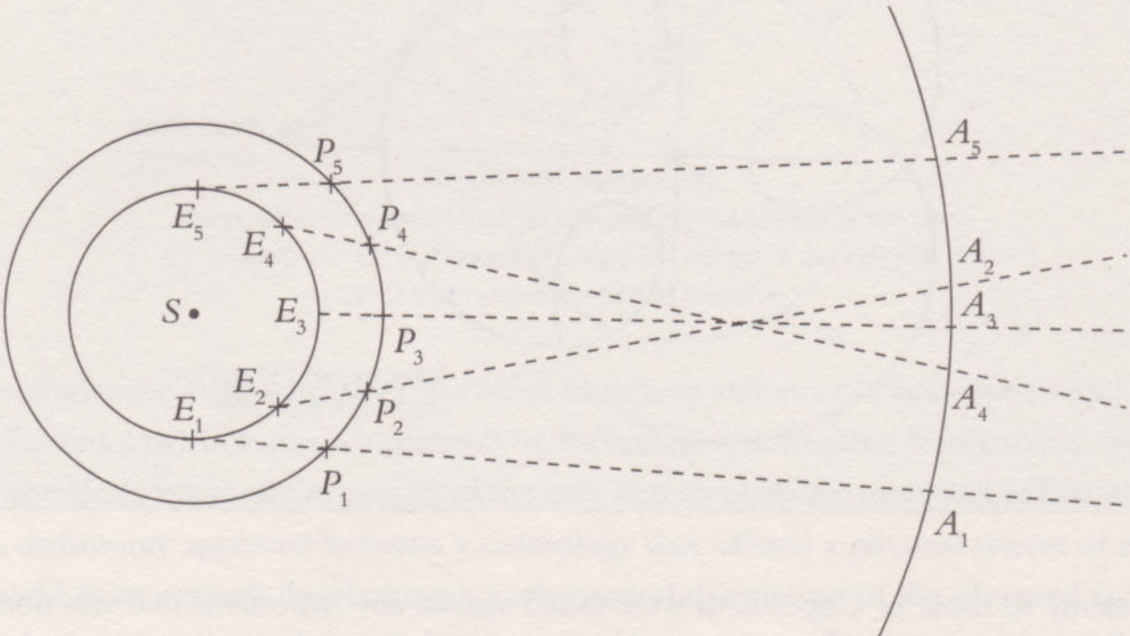


Ptolemy to those of Copernicus were raised against any hypothesis that was non-geocentric. They were essentially two: the first was the invincibility of the objections against the motion of the Earth on the grounds of physics; and the second was the disproportionate size of its universe, which was a result of the inability to observe the stellar parallax.

The new physics was born precisely from the need to answer the physical objections that the scholars of the epoch raised against Copernican astronomy. These objections were, fundamentally, identical to the arguments put forward by Aristotle

and Ptolemy to deny the Earth's motions, and were that such a motion would produce the following phenomena. Firstly, objects not attached to the Earth would be flung off by the centrifugal force developed by the extraordinary speed of the rotational movement. Secondly, all bodies not linked to the Earth, or temporarily separated from it, such as clouds, birds, objects tossed into the air, etc., would be left behind as a consequence of the Earth's movement. So, a stone falling from a tower would not fall by its side; a object thrown vertically into the air would not fall back onto the same spot, and so on.

From the Copernican perspective, the appearances were explained on the basis that the Earth was in motion. This was a new vision that broke away from the long-held geocentric tradition. Once it was seen that the Copernican system could potentially have a real basis, particularly from 1609, when Galileo Galilei (1564-1642) began to use the telescope to carry out systematic studies of the sky, the search was intensified for a physics that could be applied with equal validity to an Earth in motion and to all parts of the universe.



In the Copernican system, the retrogradations are explained by a question of perspective, as the Earth, in its motion around the Sun, overtakes a planet that is further away or is overtaken by an interior one. A planet (P) *is seen from the Earth* (E) *on the starry background at* A.

Thanks to the contributions of great figures such as Galileo and Johannes Kepler (1571–1630), among others, the new physics finally became firmly established by Isaac Newton (1642–1727) in his work *Philosophiæ naturalis principia mathematica* (Mathematical Principles of Natural Philosophy), published in 1687.

In the beginnings of western science the sky was observed and mathematical models were built enabling quantitative forecasts to be made of the future positions of stars and planets. In the next chapter we shall look at how these detailed observations were used to draw up calendars and measure time.

Chapter 3

Measuring Time

We humans live in space, but we also live in time. We physically move around the environment surrounding us, and we also move through the time period in which we are submerged. For that reason, since the very beginnings of civilisation, from when life became at last minimally structured, the world's peoples took care to organise not only their territories but also their time. In agricultural societies in which the conditions for sowing and harvesting were linked to the seasons, it was of particular importance to establish a common system of measuring time that would allow events and periods to be fixed in a way that was efficient and could be shared with others.

Observation of the natural cycles led to the study of the position of the stars and the motions of the planets, including the Sun and the Moon, which contributed to the development of astronomy. A clear correspondence was quickly established between the natural cycles and the celestial cycles, as the seasons were connected with the solar motion through the ecliptic, while the tides were linked to lunar motion. Observational reasons such as these were those that led to the creation of the two universal patterns around which different cultures and civilisations organised their calendars – lunar calendars and solar calendars.

An ancient challenge

The impossibility of harmonising the natural cycles

Calendars, a system of dividing time into days, months and years, provide a bridge between cosmic time and the time lived by each individual, but they also create a social time, a time that is understood and followed by all the community governed by one particular calendar. A calendar fulfils two functions: it provides a rhythm for time and a purpose for measuring it. It creates a structure and differentiates, for example, the workdays from the rest days, and sets traditions that create bonds between members of the community.

All calendars are based on observing the movements of celestial bodies and as a unit of measurement use some cycle that is observable by everyone. Years, months and

days have approximate lengths according to how they are defined, as the following table shows:

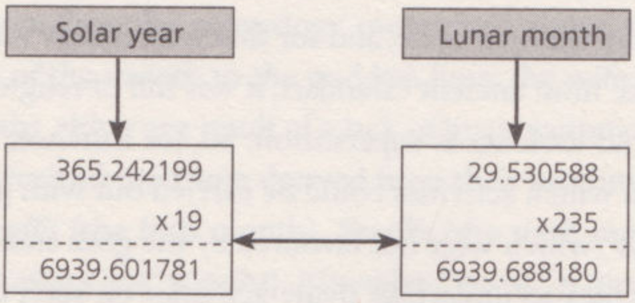
Cycle	Definition	Approximate length
Sidereal year	The time the Sun takes to be in the same position with respect to a star.	365 days, 6 h, 9 min, 9 s. (365.256363 days)
Solar or tropical year	The time for Earth to complete a revolution of the Sun, between two equinoxes.	365 days, 5h, 48 min, 46 s. (365.242199 days)
Lunar year	Period of 12 lunar months	$29.5 \cdot 12 = 354$ days
Lunar month	Interval between two new moons	29 days, 12 h, 44 min, 3 s (29 days, 6 h. to 29 days, 20 h.)
Day	Interval between two rises or settings of the Sun, or two rises or settings of the Moon.	From 23h, 59 min, 39 s to 24 h, 0 min, 30 s.

The epact (from the Latin *epactae*, *-ārum*, which came from the Greek ἐπακταί, 'added' or 'intercalated') is the number of days by which the solar year exceeds the common lunar year of twelve lunations. The epact is used for calculating the date of Easter, which is the Sunday following the first full moon after the spring equinox.

For the calendar to be practicable, time has to be reduced to whole numbers, and so, for instance, the day has 24 hours. Each society makes its choice and deals with the difficulties that brings. They decide to establish the calendar based on the Moon, the Sun or some other star. Once the choice is made, they simplify the calculations by using averages. Calendars are approximations calculated in an attempt to harmonise the complicated astronomical cycles.

Meton's cycle

The Greek astronomer Meton of Athens (5th century BC) is known for having developed an efficient system for adapting the lunar calendar to the solar year. Meton noted that 19 solar years corresponded to 235 lunar months. As 19 lunar years were 228 months ($19 \cdot 12 = 228$), the secret was in adding 7 months intercalated to the 19 lunar years ($[19 \cdot 12] + 7 = 235$) so as to have the lunar and solar calendars aligned. So, in a cycle of 19 years there will be 12 years of 12 months and 7 years of 13 months. The Athenians were so impressed by this discovery that they engraved the Meton cycle in letters of gold in the Temple of Athens for the Olympic Games of 432 BC.



In the 4th century, the Jews made use of the cycle of Meton and established a lunisolar calendar, that is, a calendar adapted to the cycles of both the Sun and the Moon. They needed it in order to adapt their traditional lunar calendar to the rhythm of the seasons. The Passover festival commemorated the Exodus from Egypt and was supposed to coincide with the spring festival. When the months became too misaligned with the seasons, the barley needed for the Passover rites failed to grow in the month of Passover. To remedy this the Sanhedrin found a practical solution by doubling the last month of the year. As the solar year has 11 days more than the lunar year, to make them even it was arranged to add a month each two or three years, following the series 3, 6, 8, 11, 14, 17 and 19. Thus the Passover festival (Pesach) was made to fall always in the first month of spring: Nisan. The calculation of the years to which a month needed to be added is similar to the one we shall see further below in the Chinese calendar.

The Gregorian calendar

The first Roman calendar

It is said to have been Romulus himself who, perhaps based on an ancient Etruscan calendar, established the first Roman calendar. It began in spring and lasted 304 days. As that year was too short for the natural passing of the seasons, it became necessary to add a set of days on the end so as to bring it into line. To count the passing years, the year Rome was founded, 753 BC, was taken as the beginning of the ancient Roman calendar. After the number the acronym *a.u.c.* was added to represent *ab urbe condita,* meaning 'since the founding of the city'. So, for example, the year 50 *a.u.c.* in the Roman calendar corresponds to the year 703 BC in our calendar. It is simply a difference in the origin of the system.

This calendar had a basis that was lunar – the months stem from it – but the aim was to adapt it to the seasonal cycle, and for that reason days were added to bring it into alignment. Like most ancient calendars, it was full of religious aspects and ideas that today we would look on as superstition. So, for instance, there were *dies fasti*, 'auspicious days', in which activities could be carried out with good omens, and *dies nefasti*, 'ill-fated days', which were not favoured by the gods and, therefore, it was not considered a good idea to undertake many activities on such days. The distinction between 'auspicious' and 'ill-fated' evokes the distinction we might make between 'leisure' and 'business'. There were also special days, such as the *nefastos partem diem*, which were partially ill-fated days, that is, until the priests offered sacrifices in the temple. There was even one day, *quando stercus delatum fas*, dedicated to cleaning the temple of Vesta, the day being considered ill-fated until the job had been done and the refuse taken out of the door known as the *Porta Stercolaria*. This name, *Fastos*, is the title of a long poem by Publius Ovidius Naso (43 BC–17 AD) in which he describes the months of the year – which was now the Julian twelve-month year – pointing out the special days, their feast days and the rites and legends associated with them. It is a magnificent work for learning at first hand about the ancient Roman calendar.

Some of the months of the Roman calendar had proper names that were a reference to divinities and were the origin of the names of our months. *Martius*, from which our name March derived, was the first month of the year and was dedicated to Mars, the god of war and, according to legend, the father of Romulus. During the Republic, from the 5th century BC onwards, the maximum authority in Rome was held by the consuls, who were elected each year and who began to exercise their mandate during the month of March.

The month of *Aprilis*, which was the origin of our April, was the second month in the ancient Roman calendar and was dedicated to Venus. The name's etymology is not very clear. Some sources claim it derived from *aperio*, which means 'open', as it is the month in which nature opens out in all its splendour, while others see its origin in the word *aper*, a wild boar, which was venerated by the Romans. Perhaps lacking etymological value but with undeniable poetic sense, Ovid even links *aprilis* to *aphros* (ἀφρός), which in Greek means 'sea foam', which is the mythical origin of Aphrodite.

Our month of May comes from the Roman month of *Maius*. This month is believed by some to have been dedicated to the elderly (*majors*), and that the following month, June, may have been dedicated to the youth (*iuniors*). Other opinions link the name to that of the goddess *Maia*, one of the Pleiades and the mother of Mercury.

The fourth month in the old Roman calendar was *Junius*, the forerunner of our June. Notwithstanding the etymology mentioned above, there are those who attribute the name of the month to the goddess Juno, the wife of Jupiter.

The other months, either as a result of a lack of imagination or an excess of desire for order, initially received the name derived from their position in the sequence of that first year: *Quintilis* (the fifth month), *Sextilis* (the sixth month), *September* (the seventh month), *October* ('*octo*' is eight), *November* ('*novem*' is nine), *December* ('*decem*' is ten). The months of *Martius*, *Maius*, *Quintilis* and *September* had 31 days, and the other six months had 30 each. This made up a strange year of 304 days.

The warrior Romulus was succeeded by a wise man, the king-priest Numa Pompilius (715 BC–672 BC). He arranged the group of days that remained at the end of the year into two new months, *Januarius* and *Februarius*, thus bringing the months up to twelve. The month *Januarius* was dedicated to the god Jano, a divinity who had two faces and could therefore look in two directions at the same time. He

OVID'S *FASTI* AND THE CALENDAR

Ovid spoke with sarcasm of Romulus's ten-month year: "When the founder of the city was setting the calendar in order, he ordained that there should be twice five months in his year. To be sure, Romulus, thou wert better versed in swords than stars." And later, in a critical comment on the lunar origin of the month, he writes: "Hence through ignorance and lack of science they reckoned lustres, each of which was too short by ten months. A year was counted when the moon had returned to the full for the tenth time".

The 304-day years also appeared in Ovid's text and are justified in a curious way: "Yet, Caesar, there is a reason that may have moved him, and for his error he might urge a plea. The time that suffices for a child to come forth from its mother's womb, he deemed sufficient for a year." One must admit, however, that this strange replacement of the habitual astronomical arguments for setting the length of a year with arguments of such a human nature has a certain charm to it.

Publius Ovidius Naso.

was the god in charge of opening and closing doors, which is the reason why he is sometimes shown with a key in his left hand; in fact, the word *janua* means door. The month *Februarius* was dedicated to the dead and the purification rites; the name appears to derive from the word *februa*.

Januarius was linked to other important events, such as the fact that, as time went by, this month came to signify the starting point of the period of office of a consulate. In the year 154 BC, the Celtic-Iberian city of Segeda, near to the present-day city of Calatayud, in the Hispania Citeriora region, decided to change its area of influence and its fortifications with the aim of forming a group with neighbouring communities. The Roman Senate considered this to be a threat to the peace and forbade it. As the inhabitants of Segeda failed to obey the orders from Rome, war was declared on them, with Quintus Fulvius Nobilior being entrusted with a force of almost 30,000 soldiers. So that he could arrive in Hispania early enough for the operations not to drag on too far into the middle of winter, that year – for the first time – the consul was sworn in on the 1st of *Januarius*, and they continued to do so from then onwards. This meant that the year 154 BC ended with two months fewer than it should have done. Consequently, the original distribution of the months was moved two units, so that *Quintilis* came to take seventh place; *Sextilis*, the eighth; *September* moved to the ninth place; *October* took the tenth place; *November* went to eleventh place; and, finally, *December* ended up in twelfth place.

Numa Pompilius believed that even numbers brought bad luck and was not willing to allow months with an even number of days, so he changed all the months that Romulus had given thirty days (*Aprilis*, *Junius*, *Sextilis*, *October*, *November* and *December*), to 29 days, the others remained the same with 31. To the new *Januarius* he gave 29 days, and only *Februarius* was given the sinister even number of 28 days. That gave a year with 355 days, which still did not fit that well with the natural periods. To correct this misalignment, every two years a series of supplementary days was intercalated, of 22 and 23 days alternatively, which was called *mercedonius*, a term which comes from *merces*, meaning salary or reward, as during that period of time it was the custom to make a payment to the slaves. Consequently, a period of four years in this calendar would have the following distribution of days:

First year:	355 days.
Second year:	377 days.
Third year:	355 days.
Fourth year:	378 days.

This gives a total of 1,465 days in four years. Bearing in mind that in our calendar four years correspond approximately to 1,461 days, we can see that Numa Pompilius's year was too long by some four days. The intercalation of the *mercedonius*, along with the establishment of the auspicious and ill-fated days, the days of assembly, market-days and so on, was the responsibility of the priest caste. Having the 'monopoly of time' was a great advantage and to maintain that control the rules for setting the calendar were kept secret until 304 BC, when Gnaeus Flavius announced in the forum a list with the days for the administration of justice. This action represented the beginning of the secularisation of the calendar, which started to move away from being a matter reserved for the priests and patricians, to become more open, and to be under the control of what we understand today as civil society.

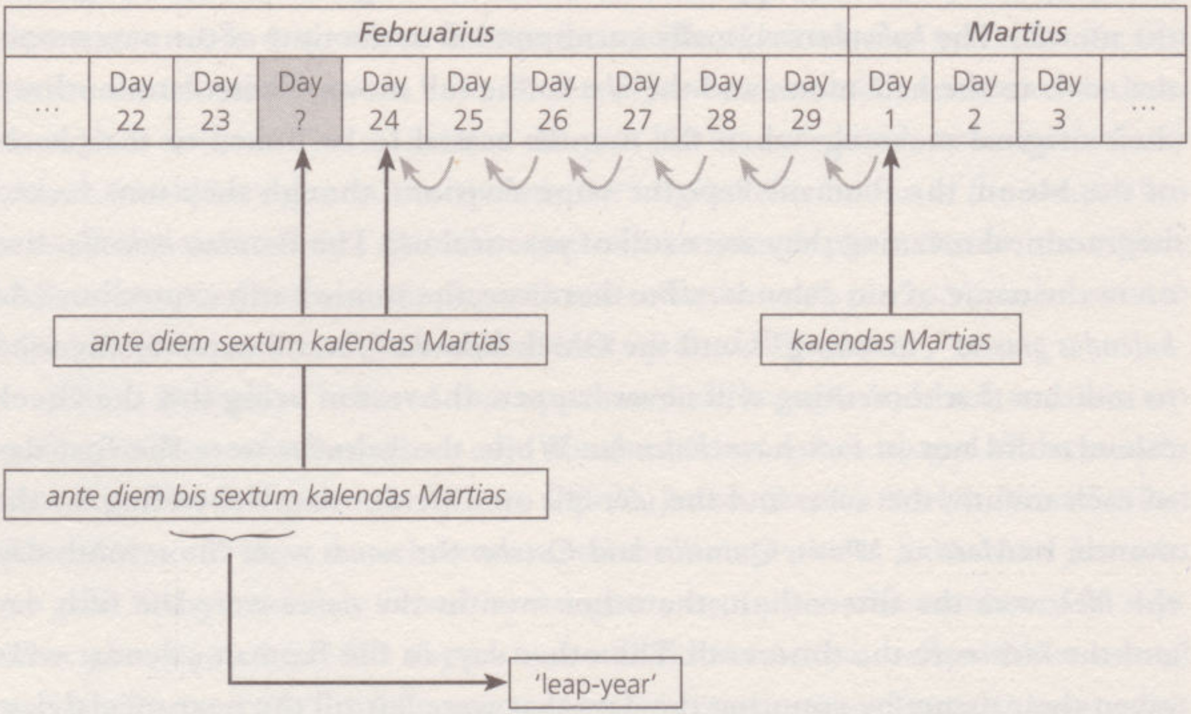
In the Roman calendar, each month has three special dates, the *kalendas*, the *nones*, and the *ides*, which appear to have been a vestige of the lunar origins of the months. The *kalendas* originally corresponded to the time of the new moon, the *nones* to the half-moon and the *ides* to the full moon. These dates outlived their original meaning: when the months ceased to be linked to the phases of the Moon, the Romans kept the same divisions; though they now lacked astronomical meaning they were still of practical use. The Roman *kalendas* live on in the name of our calendars. Furthermore, the ironic Latin expression "*Ad kalendas graecas*", meaning "Until the Greek *kalendas*", is still occasionally used to indicate that something will never happen, the reason being that the Greek calendar did not in fact have *kalendas*. While the *kalendas* were the first day of each month, the *nones* and the *ides* fell on different days depending on the month: In *Martius*, *Maius*, *Quintilis* and *October* the *nones* were the seventh day, the *ides* were the fifteenth; in the other months the *nones* were the fifth day and the *ides* were the thirteenth. The other days in the Roman calendar were given their names by counting the days that were left till the next special date (*kalendas*, *nones* or *ides*).

The Julian reform

In the year 46 BC, Julius Caesar rearranged the Roman calendar under the guidance of the astronomer Sosigenes of Alexandria and set the length of the year at 365 days and a quarter. As Sosigenes well knew, the Egyptian calendar had a duration of 365 days, and so only corresponded to the year's real length (365.242199 days) as far as the units were concerned, and was therefore out by 0.242199 days each year. This

came to 0.968796 days – almost a full day – every four years. Julius Caesar added that day every four years.

He set the length of the months alternatively at 31 and 30 days, except for the month of *Februarius* which, in normal years, was left at 29 days. Every four years a day was added to *Februarius* and intercalated between the 23rd and 24th of the month, which therefore had 30 days. The names of the days in the Roman calendar were determined by the number of days that remained until the next notable date (kalendas, nones or ides). Thus, the 24th of February was the *ante diem sextum kalendas Martias*, and to refer to the intercalary day they used the prefix *bis*. It was therefore known as *ante diem bis sextum kalendas Martias*. From this term's words *bis sextum* derived the name *bissextile* [leap year] for those special years. See the table:



When Julius Caesar died in 44 BC, in his honour the Roman Senate decided to replace the name of the month of his birth, *Quintilis*, with *Julius*. Likewise, in the year 8 BC the name of the month *Sextilis* was replaced by that of *August* in honour of Caesar Augustus. This raised a question of protocol: how could it be that *Julius* had 31 days and *August* only 30? The solution was easy – add one day to *August* and, of course, take it off *Februarius*, which was always the loser in all the changes, perhaps because it had been the last month... and the month of purification. As now there were three consecutive months with 31 days (*Julius*,

August and *September*), it was decided to transfer one day from *September* to *October* and one from *November* to *December*. These changes are summed up in the following tables:

7th century BC

Martius	*Aprilis*	*Maius*	*Junius*	*Quintilis*	*Sextilis*	*September*	*October*	*November*	*December*	*Januarius*	*Februarius*
31	29	31	29	31	29	31	29	31	29	29	28

Year 46 BC

Januarius	*Februarius*	*Martius*	*Aprilis*	*Maius*	*Junius*	*Quintilis*	*Sextilis*	*September*	*October*	*November*	*December*
30 + 1	30 − 1 Normal: 29 Leap-year: 30	30 + 1	30	30 + 1	30	30 + 1	30	30 + 1	30	30 + 1	30

Year 44 BC

						Julius					

Year 8 BC

							August				
	−1						+1				
								−1	+1	−1	+1

Present situation

Januarius	*Februarius*	*Martius*	*Aprilis*	*Maius*	*Junius*	*Julius*	*August*	*September*	*October*	*November*	*December*
31	Normal: 28 Leap-year: 29	31	30	31	30	31	31	30	31	30	31

Thus was our current distribution of the days of the month established, just as we learnt in our days at infant school with the well-known rhyme: "30 days has September, April, June and November, of 28 there is but one, and all the rest have 31...."

In ancient Rome, the years were counted starting from the year the city was founded. The retiring Emperor Diocletian tried to have the count starting from the date he took the throne (284 AD). Finally, the origin of our enumeration system for the years was due to Dionysius Exiguus, who set the starting point at the year of Jesus Christ's birth, with a probable error of 4 or 5 years. Under the new enumeration system, this change came about in the year 531 AD.

By adding a day every four years the reform made by Julius Caesar established the length of the civil year at 365.25 days, thereby achieving a closer approximation to the year's real length of 365.242199 days. The difference between these values still leaves an error of 0.007801 days each year. The difference may seem very small, but every 400 years it comes to 3.1204 days. By the 16th century the deviation had reached 10 days and it became necessary to make another reform of the calendar, leading to the establishing of what is known as the Gregorian calendar – the one we use nowadays.

The Gregorian reform

The Gregorian calendar was introduced by Pope Gregory XIII in 1582 following studies by a commission presided over by the Jesuit Christopher Schüssel (Clavius), a mathematician and astronomer. Two changes stand out. The first was to keep multiples of four as leap-years but to eliminate three leap years every 400 years; so, years that are multiples of 100 cease to be leap years unless they are multiples of 400. The second one eliminated the 10 days' lag that had built up

The German Jesuit Christopher Schüssel, who, together with the Italian doctor and astronomer Luigi Lilio, was a notable member of what was known as the 'Calendar Commission'.

ANECDOTES ON THE INTRODUCTION OF THE GREGORIAN CALENDAR

Russia did not accept the Gregorian reform until the arrival of the Soviets. The adoption of the new calendar was done in such a way that 1 February 1918 became 14 February 1918. Curiously, that meant the October Revolution was commemorated in November in the USSR. The revolution had begun on the 25th and 26th of October under the Julian calendar that was in force in Tzarist Russia; when transferred to the Gregorian calendar, those dates fell on the 7th and 8th of November. Shakespeare and Cervantes died on the same date, 23 April 1616, but not on the same day. Cervantes died on the 23 April 1616 according to the Gregorian calendar, which had been in force in Spain since 1582, while Shakespeare died on 23 April 1616 under the Julian calendar, which continued to be used in England till 1752.

Some biographies give Isaac Newton's year of birth as 1642 and others give it as 1643. It is no error: Newton was born on 25 December 1642, in accordance with the Julian calendar, which corresponds to 4 January 1643 under the Gregorian.

In England, the change of calendar and the elimination of days led to some unrest. It is curious to note the reference in the painting *An Election Entertainment,* by William Hogarth, which shows an activist injured after stealing a banner from a conservative demonstrator. The banner displays the words: "*Give us our eleven days*". The picture was painted three years after the change of calendar.

Detail from the painting An Election Entertainment, *by William Hogarth, in which the sentence "Give us our Eleven Days" can be seen marked with a circle.*

since the introduction of the Julian calendar: Thursday, 4 October 1582 (Julian date) was followed by Friday, 15 October 1582 (Gregorian date). This change was adopted from the outset by Spain, Italy, France and Portugal. Other countries adopted it progressively: Britain in 1752, Finland in 1918, and Turkey in 1926, for instance.

The Islamic calendar

The Hegirian calendar is the official calendar in the Islamic world. The Caliph 'Umar ibn al-Khattab decreed that the Muslim era would begin on 16 July 622, the date that corresponded to the Hegira (from the Arabic *hiÿra*, 'flight') of the Prophet Muhammad to the city of Medina. The Islamic calendar is lunar, based on a cycle of 12 lunar months. Months of 30 and 29 days alternate (odd months have 30 and even ones have 29) forming years of 354 or 355 days.

Muhammad claimed that Allah had put the Moon in the sky so that time could be measured and for that reason forbade any change to the names of the year's twelve moons, which are partly based on agricultural-livestock themes and partly on religious issues. As we shall see, the twelve moons do not exactly correspond with the twelve months in the Gregorian calendar, so we shall call them first month, second month, etc., instead of January, February etc.

The year begins in *Muharram* ('sacred month'). Its name derives from *haram*, 'prohibited', and in this month it is not allowed to wage war; some Muslims fast the entire month, the same as at *Ramadan*. It has 30 days, the first one called *R'as as-Sana*. Though this day has no particularly significant religious connotations, many Muslims use the date to remember the life of the Prophet Muhammed and the Hegira.

Safar, the second month, has 29 days. Its name comes from the Arabian word *sufr* ('yellow'), as originally it was in autumn, 'when the leaves turn yellow'. In any case, this month was considered the most ominous in the calendar as, according to tradition, it was then that Adam was expelled from the Garden of Eden.

Rabi' al-Awwal is the third month. During this month, all the world's Muslims celebrate the *Mawlid* (birthday of the Prophet). The great majority of Sunni Muslims believe that the exact birth date of Muhammed is the twelfth of the month, while the Shias believe he was born in the early hours of the seventeenth day.

Rabī' al-Thānī is the fourth month in the Muslim calendar. It is also known as *Rabī' al-Ākhir*. *Jumada al-Wula* is the fifth, and *Jumada al-Thania* the sixth.

Rajab (from the Arabic *Raândab*, transcribed sometimes as *Rayab*), is the seventh month and has, like all the odd-numbered months, 30 days. Its name is a reference

to respect. Pre-Islamic Arabs held the month of *Rajab* in great esteem and during this month, as in *Muharram*, war was banned. A quotation attributed to Muhammad claims that he who fasts at *Rajab* will drink from the fountain of life in paradise. Devoted Muslims fast on the first Friday of the month.

Sha'aban (from the Arabic *Ša' bān*; which can also be transcribed as *Sha'bán*, *Chaabán*, etc.) is the eighth month and has 29 days.

Ramadan (from the Arabic *Ramadān*) is the ninth month and is well-known throughout the world for being the month in which Muslims fast every day from dawn till dusk. The word Ramadan is normally used in English to refer to the actual fast, but the Arabic term for this is *sawm*.

DATES FOR RAMADAN IN RECENT YEARS

The Islamic calendar is lunar; the months begin when the first quarter-moon is visible after the new moon, in other words, a couple of days after the new moon. The Islamic Calendar year is shorter than the Gregorian, and for that reason the dates give the appearance of 'moving' through the Gregorian year. For example, these are the Ramadan months for recent years:

Ramadan month	In the Gregorian Calendar
Of year 1427 of the Hegira	23 September–22 October 2006
1428	12 September–11 October 2007
1429	1 September–30 September 2008
1430	22 August–19 September 2009
1431	11 August–10 September 2010
1432	1 August–29 August 2011
1433	21 July–19 August 2012

Working out exactly when Ramadan begins is important so as to ensure compliance with the month's religious obligations. Many Muslims insist on following the tradition of marking the beginning of Ramadan by sight, that is, by watching the skies for the first crescent moon after the new moon. Others let themselves be guided by the date and time calculated in advance for each zone, or they wait for an official announcement.

Shawwal (from the Arabic *Šawwāl*) is the tenth month. Its name means 'the pairing of the animals'. *Du l-qa'da* (from the Arabic *Dū l-qaqda*, also transcribed as *Dhu al-qaada*, and others) is the eleventh month and has 30 days; it is associated with rest. *Du l-hiyya* (from the Arabic *Dū l-hiândŷa,* also written as *Dhu al-hijja*) is the twelfth month. Its name means, literally, "the one of the pilgrimage", as it is the time of the year when the Muslims carry out the *haŷŷ*, the pilgrimage to Mecca. It lasts for 28 days in the years known as simple years, and one day longer in the intercalary years, which we shall look at later.

This lunar calendar contains 12 months alternating between 30 and 29 days, and takes no account of the changing of the seasons. It has 354 days, but as twelve moons or lunar months come to 354 days, 8 hours, 48 minutes and 38 seconds, the result is that at the end of the twelve months the new moon of the following year takes a little time to arrive, reaching a disparity of some eleven days after 30 lunar years. Thus, as Muhammad had forbidden the practice of adding additional months and the years based on lunar months had to be kept, some time around the year 639 Caliph 'Umar ibn al-Jattab solved the problem in an original way. He based the adjustment on lunar cycles of 30 years. In this period there would be 19 years of 354 days (*simple years,* with 6 months of 30 days and 6 months of 29 days: $6 \cdot 30 + 6 \cdot 29 = 354$) and 11 years of 355 days (*intercalary years,* of 7 months of 30 days and 5 months of 29 days: $7 \cdot 30 + 5 \cdot 29 = 355$). Because every 32 lunar months, the Moon is approximately one day behind, so an extra 11 days have to be spread over 30 years, the Caliph formed cycles of 30 years (360 lunations in the Babylonic tradition) and added one day to every year of the series of 30 years, which in the sequence of months accumulated until then contains a number of months which is a multiple of 32. In other words, if the successive multiples of 32 (months) are calculated the result is:

$$32, 64, 96, 128, 160, 192, 224, 256, 288, 320, 352.$$

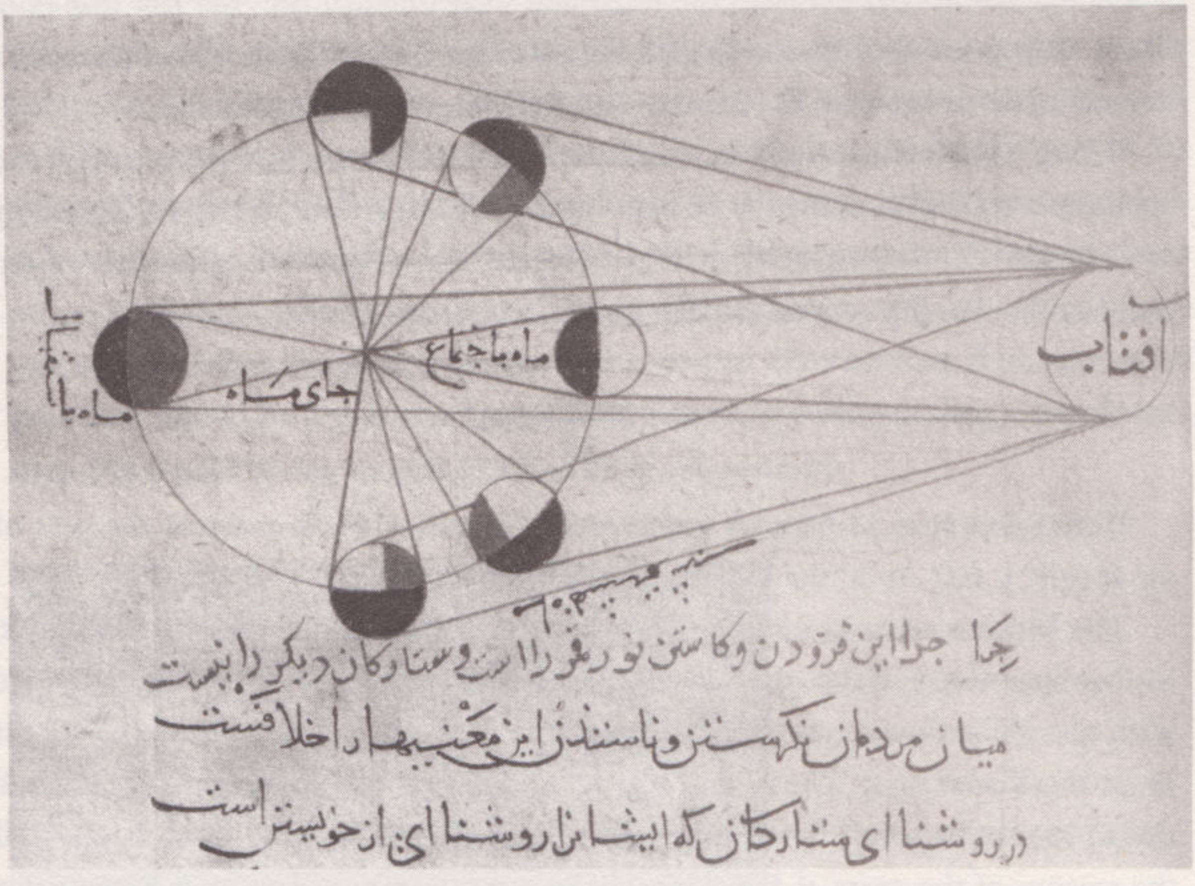
If it is now divided by 12, to find out to which year to add the day outstanding, the result is:

32/12 = 2,6667	64/12 = 5,3333	96/12 = 8
128/12= 10,6667	160/12 = 13,3333	192/ 12 = 16
224/12 = 18,6667	256/12 = 21,3333	288/12 = 24
320/12 = 26,6667	352/12 = 29,3333	

In the following series of 30 years, the years to which a day is added (in the last month) is shown in bold:

1	**2**	3	4	**5**	6	7	**8**	9	**10**
11	12	**13**	14	15	**16**	17	**18**	19	20
21	22	23	**24**	25	**26**	27	28	**29**	30

Although there is some discrepancy between Muslim calendarists, for example, on whether the intercalary year is 7 or 8, in any case, whether the year has 354 or 355 days, it never comes to the 365 or 366 days of the Gregorian calendar, so 33 Muslim years ($10{,}631 + 354 + 355 + 354 = 11{,}694$ days) are the equivalent of 32 Gregorian years ($365 \cdot 32 = 11{,}680 + 8 = 11{,}688$ days).



A drawing by the Persian mathematician and astronomer Al-Biruni (973-1048) showing the lunar phases.

This stratagem allows the Muslim lunar month calendar to coordinate with the years and the months with the Moon's cycle. In any case, every cycle of 30 years falls 16 minutes 48 seconds short of the appearance of the new moon introducing the following cycle, but as that amount is such a small value – 33 1/10 seconds per year

– 2,570 years would be needed to lose a full day with respect to the Moon, which is of little importance compared to the Gregorian calendar, which needs to eliminate 3 leap-years every 400 years.

The days are divided into groups of seven, without break, not even going from one year to the next, as happens in the Gregorian calendar; their order is as follows: *al-Ahad* ('the First'), Sunday; *al-Ithnayn* ('the Second'), Monday; *al-Thalaathaa* ('the Third'), Tuesday; *al-Arba'a* ('the Fourth'), Wednesday; *al-Khamis* ('the Fifth'), Thursday; *al-Jumua'a* ('the Meeting'), Friday – the name stems from the fact that it is the holiday when collective prayers are held in the mosques, and *as-Sabt* ('the Seventh' or 'the Sabbath'), Saturday.

The day begins with the setting of the Sun, and the month begins some two days after the new moon, when the crescent moon begins to appear. If we look at the difference in days between the lunar and solar calendar, and the fact that both begin their years on different dates, it is easy to see how difficult it is to establish a correspondence between the Muslim and the Gregorian calendars.

There exist tables showing correspondences of years, but for a quick and approximate calculation use can be made of the following formula which, depending on whether G (of Gregorian) or H (of Hegira) are worked out, gives the way to translate the Muslim year to the Gregorian or vice versa:

$$G = H\frac{32}{33} + 622.$$

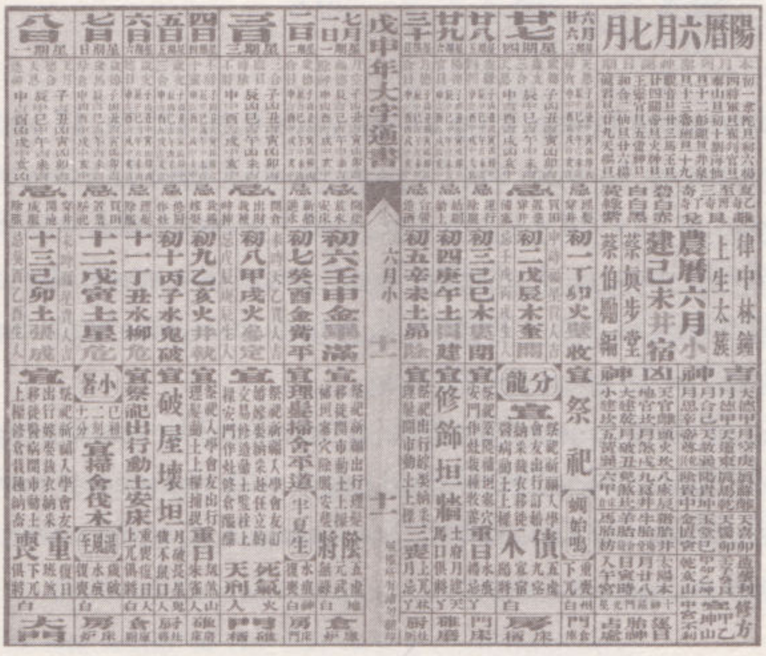
Remember that 33 Muslim years are 32 Gregorian years, and that the year of the Hegira is 622 AD in the Gregorian calendar.

The Muslim calendar contains two periods of time that stand out from the rest. Muhammad decreed that the whole month of *Ramadan* should be for penitence with the obligation to abstain from eating and drinking from sunrise to sunset or, as the Koran says: "Until the difference between a white thread and a black thread cannot be seen". Since, in the lunar calendar, the months move through all the seasons of the years, if *Ramadan* falls in summer, when the day is longest (i.e. with more hours of daylight), combining work and fasting is much harder than if Ramadan falls in winter when the day is shorter. The second notable date is the month of *Du l-hiyya* when pilgrimages are made to Mecca, which is compulsory at least once in their lifetime for all Muslims who have the means and are able to do so.

The Chinese calendar

The first astronomers in ancient China found that for counting and recording days the natural cycles were indispensable, and from very early times carried out observations of the solar and lunar cycles. Two oracle bones from the Shang dynasty show that around the 14th century BC the Chinese had already determined the solar cycle at 365 days and a quarter, and the lunar cycle at 29 days and a half. Around the year 104 BC, they managed to calculate the length of the year at 365.2502 days through observation and by measuring the shadows cast by a gnomon. In the 5th century AD the mathematician and astronomer Zu Chongzhi calculated it at 365.2428 days, with an excess of just 52 seconds over the true value (365.2422).

Instead of letting themselves be guided by the ecliptic, as the Greeks did, the Chinese astronomers observed the passing of the stars over the meridian and by using this stellar reference divided the celestial sphere into segments, in a similar fashion to the Mesopotamians' division of the zodiac. If we take 12 segments, each of them is equivalent to an interval of exactly 30.4375 days, the twelfth part of the circumference of 365.25 days. To this fraction of the year they linked one new moon (a lunar month), but as the Moon's cycle is 29.5308 days, it could happen that a moon could be a whole month late; this supplementary month was added to the year so as to adjust the lunar calendar to the seasons. That is to say, if in a period of time that should be a year, the Sun did not enter the corresponding constellation, then this would be the month that had to be intercalated.



An image of a Chinese calendar.

Later they discovered that 19 solar years were almost equal to 235 moons (6,939 days). This period of concurrence is the equivalent of the Meton cycle in western culture and enabled them to build a unisolar calendar. So, in 19 years they intercalated years of 12 lunar months and others of 13 lunar months (embolismal years). An additional basic rule is that the winter solstice must always occur in the year's eleventh month. In the following series the 7 embolismal years of 384 and 385 days are marked in bold; they alternate so as to make adjustments for the lunar cycle having 29.5 days:

1 2 **3** 4 5 **6** 7 **8** 9 10 **11** 12 13 **14** 15 16 **17** 18 **19**.

Why are they added in those years? It is modular arithmetic counting, similar to the counting that Muslims used to make their calendar fit. In the Muslim system a day was added to some years, while here a month is used. The Meton cycle determines that 19 solar years equal 235 moons. So, in 19 lunar years there are $19 \cdot 12 = 228$ moons, but in the same solar years, however, there are 235; so 7 moons must be intercalated. But when? In the years in which they appear. The following calculations show that the first moon must be added to the third year, as otherwise the period of three years would end up with 36 moons and in reality it should contain 37.

$$\frac{365.25 \cdot 1}{29.5} = 12.38 \text{ months}; \quad \frac{365.25 \cdot 2}{29.5} = 24.76 \text{ months}; \quad \frac{365.25 \cdot 3}{29.5} = 37.14 \text{ months};$$

Continuation of these calculations brings it up to the years 6, 8, 11, 14, 17 and 19, as shown in the series listed above. So, from the 19 years we shall have 12 years of 354 days, which come to a total of $354 \cdot 12 = 4{,}248$ days, and 7 years of 384, which come to a total of $384 \cdot 7 = 2{,}688$ days; to 3 of these last 7 years one more day is added, reaching a total of 6,939 days ($4{,}248 + 2{,}688 + 3$), which coincides with the number of days that each period of 19 years should have.

The date of the beginning of the Chinese New Year is calculated by combining the lunar and solar cycles. The Chinese New Year should begin with the second new moon after the boreal winter solstice (22 December). For example, if, at the winter solstice of 2000, the Moon had an 'age' of 7 days:

$(29.5-7)+29.5=52,$

where $29.5-7$ calculates the date of the first new moon after the solstice; 29.5 calculates the date of the second new moon after the solstice.

This means that 52 days must be counted from the winter solstice, and the beginning of the new year should be placed there; in the example, it is 12 February 2001. The new moon is the 'all black' one (when the Moon is in conjunction with the Sun) and not the moon visible for the first time (the first crescent) used in the Islamic and Hebrew calendars. The date of the new moon is the first day of a new month. The months are divided into three groups, *Meng* (first), *Zhong* (middle) and *Ji* (last), and into four seasons, *Chun* (spring), *Xia* (summer), *Qiu* (autumn) and *Dong* (winter). The names of the months are formed by combining both concepts, so *Ji-qiu* is the last month of autumn. The Chinese months are made up of three weeks, each of ten days. The days of the month are counted by their ordinal numbers and the day begins at midnight.

While western culture counts years in centuries, the Chinese use periods of 60 years, called *Jia-zi*, which have two components. The first component corresponds to celestial signs, the *Gan*, while the second corresponds to the animals of the zodiac, the *Zhi*.

Gan			
1	*Jia*	6	*Ji*
2	*Yi*	7	*Geng*
3	*Bing*	8	*Xin*
4	*Ding*	9	*Ren*
5	*Wu*	10	*Gui*

Zhi			
1	*Zi* (rat)	7	*Wu* (horse)
2	*Chou* (ox)	8	*Wei* (sheep)
3	*Yin* (tiger)	9	*Shen* (monkey)
4	*Mao* (rabbit)	10	*You* (cockerel)
5	*Chen* (dragon)	11	*Xu* (dog)
6	*Si* (snake)	12	Hai (pig)

Each of the two components is used in sequence; thus, the cycle of 60 years would be the result of the following combinations:

Year	*Gan*	*Zhi*
1	*Jia*	*Zi* (rat)
2	*Yi*	*Chou* (ox)
3	*Bing*	*Yin* (tiger)
4	*Ding*	*Mao* (rabbit)
5	*Wu*	*Chen* (dragon)
6	*Ji*	*Si* (snake)
7	*Geng*	*Wu* (horse)
8	*Xin*	*Wei* (sheep)
9	*Ren*	*Shen* (monkey)
10	*Gui*	*You* (cockerel)
11	*Jia*	*Xu* (dog)
12	*Yi*	*Hai* (pig)
	…	…
	…	…
58	*Xin*	*You* (cockerel)
59	*Ren*	*Xu* (dog)
60	*Gui*	*Hai* (pig)

The present 60-year cycle began on 2 February 1984. This means, for example, that the year *wu-yin*, year 15 in cycle 78, began on 28 January 1998. Year 20 in cycle 78 began on 1 February 2003.

In China the traditional calendar is nowadays known as the 'agricultural calendar', while the Gregorian calendar is known as the 'standard calendar' or the 'Western calendar'. The Gregorian calendar was introduced by the Jesuits in the 19th century and is now the calendar commonly used in everyday contexts. The Chinese calendar is used to mark certain traditional festivities such as the Chinese New Year or the festival known as Duanwu Jie – the Dragon Festival, also known as the Double Fifth because it is held on the fifth day of the year's fifth moon.

A revolutionary calendar

Several revolutionary movements have tried to abolish the Gregorian calendar in an attempt to break with the past and rearrange time in accordance with some new vision of the world. For instance, the French Republican Calendar had twelve

EQUIVALENCES BETWEEN THE CHINESE CALENDAR AND THE GREGORIAN

The table below shows the initial date of the Chinese calendar with respect to the Gregorian:

Chinese year	Zodiac animal	The Gregorian Calendar
4707	Ox	26 January 2009
4708	Tiger	10 February 2010
4709	Rabbit	3 February 2011
4710	Dragon	23 January 2012
4711	Snake	10 February 2013
4712	Horse	31 January 2014
4713	Sheep	19 February 2015
4714	Monkey	9 February 2016
4715	Cock	28 January 2017
4716	Dog	16 February 2018
4717	Pig	5 February 2019
4718	Rat	25 January 2020

months of 30 days, each divided into 3 *décades* containing 10 days each. Five or six days were added at the end of the year in the case of leap-years. This calendar, fruit of the ideals behind the French Revolution, was only in force from 1792 to 1804. This ambitious reform had three objectives: to express a rejection of the previous absolutist regime; to set in place secular festivities within the framework of the new society; and to rationalise all the weights and measures systems, including the system for measuring time.

The French Revolutionary Calendar was meant to be irreversible and it began at year 1. Its supporters argued that years could no longer be counted in the same way as in the period when the people were oppressed by a monarch – a period when they were not truly living. Time was opening a new book in history. The new era began on the 22 September 1792 with the defeat of the monarchy and the proclamation of the republic. By lucky coincidence, 22 September was the date of the autumn equinox. The revolutionaries saw it as a good omen: civil equality was in concord with the equality of day and night; history was returning to nature.

SPRING IN THE FRENCH REVOLUTIONARY CALENDAR

Spring began with the month of Germinal; its days and its associated images are given below. Every month was linked to a different feminine image.

Germinal (21 March–19 April):

1.	Primevère (Primrose)	16.	Laitue (Lettuce)
2.	Platane (Plane tree)	17.	Mélèze (Larch)
3.	Asperge (Asparagus)	18.	Ciguë (Hemlock)
4.	Tulipe (Tulip)	19.	Radis (Radish)
5.	Poule (Hen)	20.	Ruche (Hive)
6.	Bette (Chard plant)	21.	Gainier (Judas tree)
7.	Bouleau (Birch tree)	22.	Romaine (Lettuce)
8.	Jonquille (Daffodil)	23.	Marronnier (Horse chestnut)
9.	Aulne (Alder)	24.	Roquette (Rocket)
10.	Couvoir (Hatchery)	25.	Pigeon (Pigeon)
11.	Pervenche (Periwinkle)	26.	Lilas (Lilac)
12.	Charme (Hornbeam)	27.	Anémone (Anemone)
13.	Morille (Morel)	28.	Pensée (Pansy)
14.	Hêtre (European Beech Tree)	29.	Myrtille (Blueberry)
15.	Abeille (Bee)	30.	Greffoir (Knife)

The image for Germinal in the Revolutionary Calendar.

Like the reform of weights and measures, which brought in rational standards such as the kilogram and the metre, the new calendar was also arranged in a decimal system, with the idea of rationalising public life, and efforts were made to make it clear, precise, simple and universal. It was believed that the old system was a monument to servitude and ignorance and was full of anomalies, such as unequal months and wandering festivities. The new calendar used calculations based on the number ten and was in harmony with the motions of the celestial bodies. All divisions of time less than a month were in base 10. The twelve months all had 30 days and were divided into periods of ten days, the *décades*. Five intermediary days were left, which were placed at the end of the year plus an extra one every four years. This new system of arranging time was basically ancient Egypt's model (12 30-day months, divided into periods of 10 days, plus 5 extra days at the end of the year).

Introduced on 5 October 1793, the calendar was secular in character as it eliminated Sunday, the Lord's Day, and all saints days. If all religious symbols were to be suppressed, some other tradition had to replace them, and nature was chosen. Instead of associating the day with a saint, each day was linked to a plant, a mineral, an animal (days ending in 5) or a tool (days ending in 0). The date 25 December became the day of the dog. The months had a more poetic air to them. Autumn (suffix *-aire*) included the months of *Vendémiaire* (from the Latin *vindemia,* 'harvest'), harvester; *Brumaire* (from the French *brume*, 'mist'); *Frimaire* (from the French *frimas* 'frost'). Winter (suffix *-ôse*) had *Nivôse* (from the Latin *nivosus*, 'snowy'); *Pluviôse* (from the Latin *pluviosus*, 'rainy'); *Ventôse* (from the Latin *ventosus*, 'windy'). Spring (suffix *-al*) had *Germinal* (from the Latin *germen*, 'seed'); *Floréal* (from the Latin *flos*, 'flower'); *Prairial* (from the French *prairie*, 'meadow'). Summer (suffix *-idor*) had *Messidor* (from the Latin *messis*, 'crop'); *Thermidor* (from the Greek *thermos*, 'heat'); *Fructidor* (from the Latin *fructus*, 'fruit').

The common people were reluctant to accept the disappearance of popular festivities, the midsummer night bonfires, or the festivities in honour of patron saints. It was clear that the revolutionary calendar never managed to find its way into common culture. Distant from society and unable to penetrate the collective consciousness, the new calendar disappeared by stages. In year 8, the revolutionary festivities were eliminated. In the year 10 Napoleon Bonaparte reinstated Sunday as a day of rest with the aim of restoring relations between the Church and the revolutionary state. Finally, on the 15th of Fructidor in the year 8 (9 September 1805), the calendar was officially abolished. Two reasons were given – it was not

rational enough and it was too nationalist. The Gregorian calendar was reinstated on 1 January 1806, a little more than a year after Napoleon's coronation. Fortunately, the weights and measures system that those same ideals had promoted was more successful; we shall look at this in Chapter 5.

Chapter 4

Measuring the Earth

The precise measurement of the motions of the heavens served as a reference to set standards for measuring time, but people were also interested in discovering the shape and size of the world in which they lived. As well as having contributed to the measurement of the skies, Ptolemy also served as a reference for measuring the Earth thanks to his *Geography* or *Atlas of the World*, in which he described his era's known world. In the 15th and 16th centuries, the discoveries of the New World and other territories hitherto unknown to Europeans widened the known world and corrections were added to Ptolemy's work. Towards the end of the 17th century, more thorough measurements of the Earth were made by the triangulation method, which laid the foundations for geodesics. In regard to the shape of the Earth, a dispute arose between those who professed that the Earth was flattened at the poles and those who claimed that was flattened at the equator; this led to controversy over the Earth's measurements. It was decided to resolve this passionate confrontation by determining the value of one degree of meridian, which would be measured by two scientific expeditions travelling to latitudes that were as distant from each other as humanly possible.

The first ideas on the shape and size of the Earth

In antiquity the general belief was that the surface inhabited by human beings was flat as, except for geographical features such as valleys and mountains, that is how it appeared to be. But the Greek philosophers began to consider other hypotheses. Anaximander is attributed with having conjectured that the Earth is a cylinder, that is, a cylinder longer than it is wide and inhabited only on the upper disk and situated in the centre of the celestial sphere. He is believed to have designed a map of the Earth, which was later corrected and developed by Hecataeus of Miletus (c.550 BC–c.476 BC). The map showed the then known regions of Europe, Asia and Africa, which appeared on a disk bordered by a kind of circular river/ocean, with Greece situated in the centre. How big was the world? It is always risky making exact assessments of ancient measurements, but it is thought that the circular disk of Hecataeus' map was some 8,000 km in diameter.

Hecataeus' map from the 1st century BC.

If the Earth was flat, was it limitless or did it come to an end? Hecateaus seemed to limit it – but if it was surrounded by an ocean, why did it not spill over the top of the cylinder? Was the sky connected to the sea to make a barrier? How did the Earth hold itself up? The supposition that the Earth was flat raised worrying questions

ARISTOTELIAN ARGUMENTS ON THE SPHERICITY OF THE EARTH

Aristotle put forward arguments against the assumption that the Earth was flat. They included the argument that the stars varied in height over the horizon depending on where they were observed from. For instance, a traveller moving southwards saw the constellations situated to the south rising higher above the horizon, which indicated that the horizon formed a certain angle with respect to the horizon of an observer situated further north. Therefore, the Earth was not flat. In the same way, the shadow that the Earth casts onto the Moon during the partial phases of lunar eclipses always has a circular edge, whatever the height of the Moon over the horizon. What body other than that of a sphere can have a circular shadow in any direction?

that were difficult to answer. The Greeks proposed that the Earth was spherical and found solid arguments to defend the idea, as we saw in Chapter 2. But, once they had decided that the Earth was a sphere, how did the Greek thinkers calculate its size?

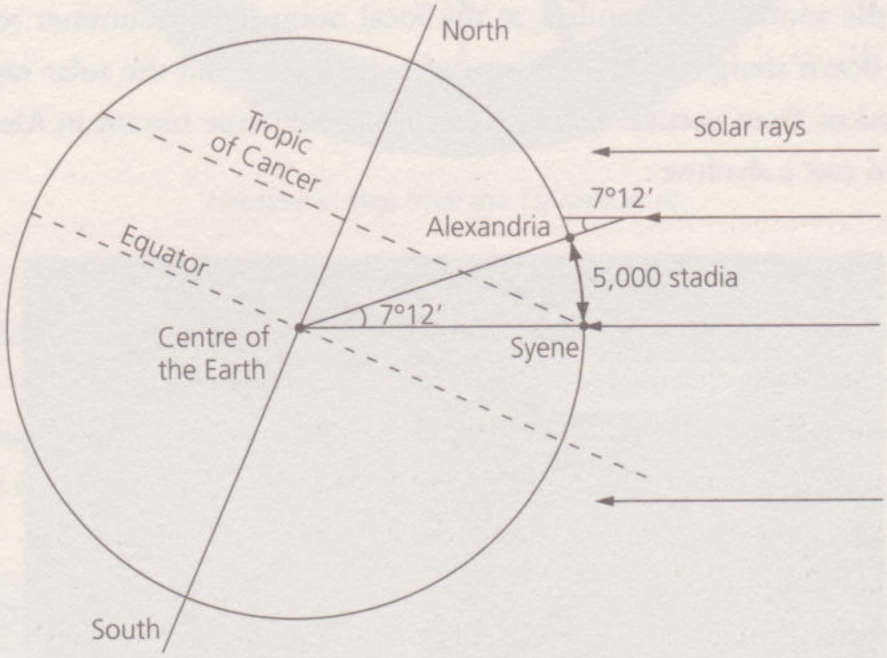
Measurement of the size of the Earth's sphere

During the Hellenistic epoch, Alexandria became the Greek civilisation's principle centre of knowledge and research thanks to its two great institutions: the Museum and the Library. It was there that the first quantitative estimation of the terrestrial circumference was made. Its author was a multi-talented Greek scholar, the mathematician and geographer Eratosthenes of Cyrene (276 BC–194 BC). As the director of the Library of Alexandria, at the time, he had access to a great amount of information stored on its papyri. He was aware that in Syene (present-day Aswan), located to the south of Alexandria, at the local noon on the summer solstice the Sun shone down straight to the bottom of a deep well, and the solar rays did not cast any shadow from vertical stakes. However, at the same instant in Alexandria, a gnomon did cast a shadow.

An engraving showing the ancient Library of Alexandria.

Supposing that the Sun is a great distance away, on reaching the Earth its rays should arrive in parallel form, and if the Earth was flat – as many people of the time thought – there should be no difference between the shadows cast by equal objects at the same time on the same day, wherever they are located. However, it was clear that the shadows were different and, therefore, the Earth was not flat.

At noon on the summer solstice in Alexandria, by using a gnomon Eratosthenes measured the angle that the Sun's rays formed with the vertical and found that it was 1/50 of a circle (7° 12'). By supposing, as he did, that the Earth is spherical (360°) and that Alexandria is to the north on the same meridian as Syene, a simple geometrical reasoning (see the diagram) enabled him to deduce that the central angle between the two terrestrial radii corresponding to Syene and Alexandria was also 1/50 of a circle (7° 12'), that is, one fiftieth of all the total circumference.



A diagram showing Eratosthenes' reasoning.

As Eratosthenes knew that the distance between the two cities was 5,000 stadia (about 800 km.), knowing that proportion was sufficient for him to calculate the circumference of the Earth. It had to be 50 times the distance between the two cities: 250,000 stadia. He in fact rounded up the result and took a final value of 700 stadia per degree, which gives a circumference of 252,000 stadia.

There have been arguments over the exact length of the stadia that Eratosthenes used. The Greek stadium is some 185 metres long, which would give a terrestrial

circumference of 46,620 km (16.3% greater than the actual circumference). However, supposing that Eratosthenes used the Egyptian stadium (he lived in Egypt), which was 157.5 m, the measurement was 39,690 km (an error of under 2%).

Eratosthenes' mathematical reasonings were rigorous, although some points should be mentioned regarding the accuracy of his measurements. Syene is not directly south of Alexandria on the same meridian, and the Sun is a disk situated at a finite distance from the Earth rather than a point of light at an infinite distance. Furthermore, in ancient times the measurements of distances on land were not reliable and were a source of experimental errors. Given the margin of error for each of the aspects concerned in his calculation, the accuracy of the size of the Earth as measured by Eratosthenes is surprising – and perhaps a fluke.

Maps of the Earth: latitude and longitude, positioning and projection

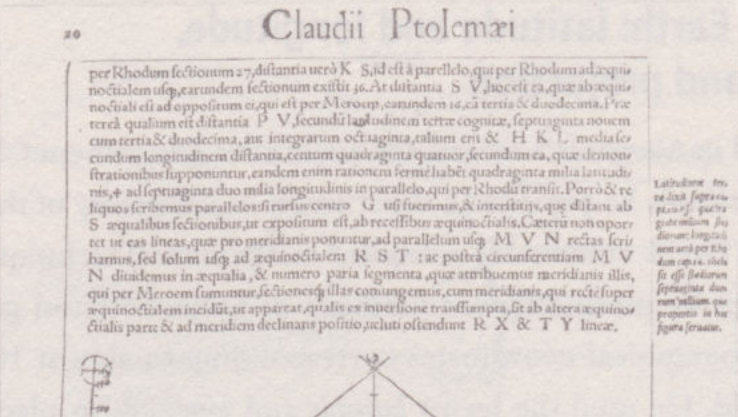
Ptolemy worked in Alexandria several centuries after Eratosthenes. By using rigorous scientific methods he provided a description in *Geography* of the whole of the world known to Greek culture. He compiled mathematical techniques for drawing up accurate maps by using different methods of projection and gathered a large collection of geographical coordinates corresponding to almost 10,000 places in the known world. He used the terms *latitude* and *longitude* to place them on the map by using a reference system of meridians and parallels. The zero meridian (his meridian 0°) was near the Canary Islands, and the zero parallel was near the equator. He situated the far north of the inhabited earth on the parallel of the semi-mythical Thule Island.

It seems that Ptolemy's measurements of the Earth were lower than the real ones due to the assumption that each degree on the equator corresponded to an arc of some 80 km, which reduced the length of the great circle to a little less than 30,000 km. Ptolemy's prestige and influence during the Renaissance encouraged sailors to dare to cross the ocean in search of the other side of the globe.

The problem of how to represent a curved surface on a map is a mathematical issue. In this respect Ptolemy also made useful contributions to cartography. Before him, Hipparchus is believed to have used a division of the Earth's circles into 360° and to have designed a grid of parallels and meridians. Hipparchus took an interest in projecting a spherical surface onto a flat map, and there are some who claim he invented the stereographic projection. Another geographer and cartographer who

had a great influence on Ptolemy was Marinus of Tyre (c.60–c.130), who was the first to use the meridian of the Canary Islands as meridian zero, and the Rhodes parallel as the starting point for latitude. He apparently proposed the cylindrical projection for drawing up maps.

To represent the Earth's surface on a plane, Ptolemy conceived of a conic cartographic projection, as well as a pseudoconic projection, and managed to vary the scale on one and the same plane. In his conic projection he showed the parallels as concentric arcs, and the meridians as straight lines converging at a point situated at the North Pole. His second projection, the pseudoconic, displayed the meridians as curved lines converging at the pole, which enabled him to show a greater extension and provide better proportions.


20 Claudii Ptolemæi

per Rhodum sectionum 27, distantia uerò K S, id est à parellelo, qui per Rhodum ad æquinoctialem usque, earundem sectionum existit 36. At distantia S V, hoc est ea, quæ ab æquinoctiali est ad oppositum ei, qui est per Meroum, earundem 16, cũ tertia & duodecima. Prætereà qualium est distantia P V, secundũ latitudinem terræ cognitæ, septuaginta nouem cum tertia & duodecima, aut integrarum octuaginta, talium erit & H K L media secundum longitudinem distantia, centum quadraginta quatuor, secundum ea, quæ demonstrationibus supponuntur, eandem enim rationem fermè habẽt quadraginta milia latitudinis, + ad septuaginta duo milia longitudinis in parallelo, qui per Rhodũ transit. Porrò & reliquos scribemus parallelos: si rursus centro G usi fuerimus, & interstitijs, quę distant ab S æqualibus sectionibus, ut expositum est, ab recessibus æquinoctialis. Cæterũ non oportet ut eas lineas, quæ pro meridianis ponuntur, ad parallelum usque M V N rectas scribamus, sed solum usque ad æquinoctialem R S T: ac postrà circunferentiam M V N diuidemus in æqualia, & numero paria segmenta, quæ attribuemus meridianis illis, qui per Meroem sumuntur, sectionesque illas coniungemus, cum meridianis, qui rectè super æquinoctialem incidũt, ut appareat, qualis ex inuersione transsumpta, sit ab altera æquinoctialis parte & ad meridiem declinans positio, ueluti ostendunt R X & T Y lineæ.

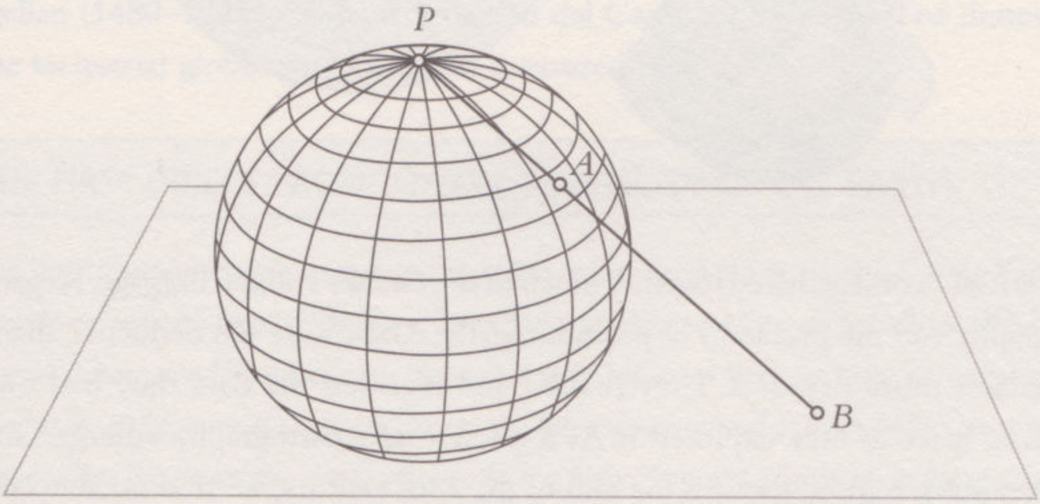
Demum propter facilem locorum ordinandorum adnotationem iterũ regulamentũ tenue faciemus, quod æquale sit longitudini G F tantũ: illudque firmabimus in G, ita, ut cũ per totam longitudinem descriptionis circũferatur, exactè rectis alterius lateris congruat meridianis, quo sectio eius per medium cadat polũ: latusque hoc in segmenta, si ad G F pertinere debent, centum triginta & unam diuidemus, aut solum in segmenta centum quindecim, si G S respiciant: numerosque adnotabimus facientes initium à sectione quæ per æquinoctialem est; à quibus & parallellos scribi intelligere possumus, ne dum meridianũ, qui in descriptiõe est, in omnia segmenta diuidimus & signamus, inscriptiones locorum, quæ super ipsum casuræ sunt confundamus. Partiemur igitur æquinoctialem in duodecim horarum centum octuaginta partes, numerosque apponemus, initium faciẽtes à meridiano occidentalissimo, ac semper regulamẽti latus ad ostensam longitudinis partem mouebimus donec per diuisionem in regulamento factam, ad positionem secundum latitudinem signatam perueniamus, ibique super unoquoque signationem debitam, codẽ quo in sphæra ostensum est modo faciemus.
Porrò

Ptolemy's conic projection, which appears in the Geography [Geographicae enarrationis libri octo] *as published in Lyon and Vienna in 1541.*

Ptolemy's conic projection was used to represent the ancient world until the discoveries of 15th-century sailors notably increased the number of known territories, meaning that this type of projection came to be inadequate for showing them all ; hence, its use became restricted to regional cartography.

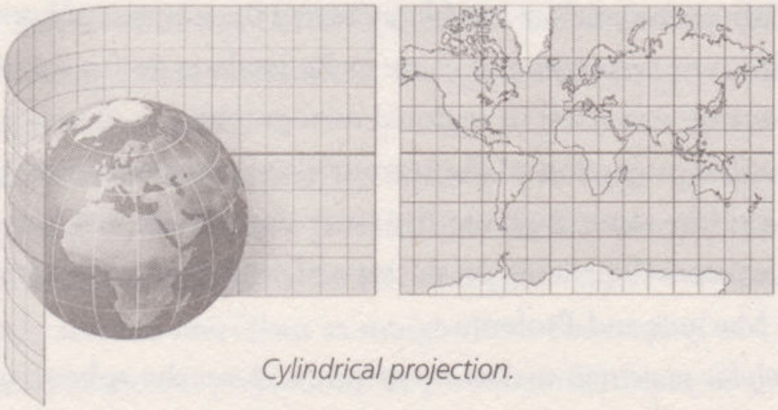
In a cartographic projection of the terrestrial sphere it is impossible to preserve the same areas and the same angles at the same time. There are, however, different possible approximations depending on the type of projection, such as those attributed to Hipparchus, Marinus and Ptolemy.

The *stereographic projection* makes every point A on the sphere, other than the pole (the focus of the projection), correspond to the point on the plane that is the intersection of the straight line PA with the plane. Reciprocally, for every point B on the plane there corresponds the unique point A, other than P, which is the intersection of the sphere with the straight line PB. Ptolemy explained this projection in his *Planisphaerium,* and he used it to project the celestial sphere onto the plane. Later, Arab scholars were to use the ideas behind the projection to construct astrolabes that enabled them to determine the position of the stars on the celestial sphere.



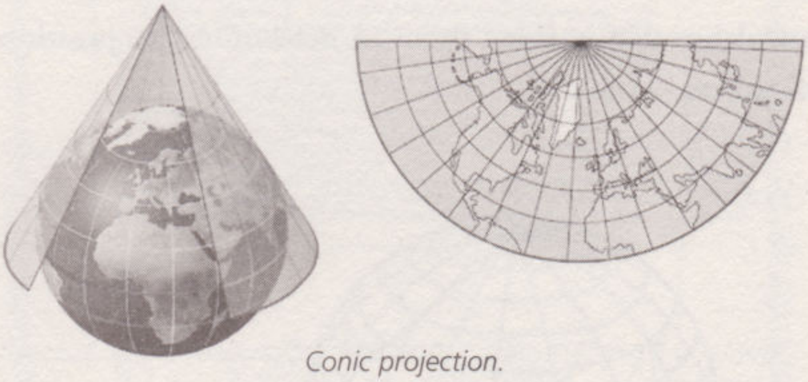
Stereographic projection.

Cylindrical projection consists of projecting the terrestrial surface onto a cylinder that is tangential to it at the equator. The resulting map displays distortions that are small near the equator and very large at the polar regions. This type of projection preserves the angles but not the areas; it increases the areas as it moves from the equator towards the poles.



Cylindrical projection.

The *conic projection* projects points on the Earth onto a cone by taking one of the poles as a focal point. The polar regions are distorted, but this projection gives greater precision in the hemisphere of the pole chosen as the focal point. The map's distortions are small along the contact parallel but greater as we move away from it.



Conic projection.

The Arab world gathered together much of the Greeks' cultural baggage. Regarding cartography and the problems of positioning, the Arabs were less deductive than the Greeks and more practical. They revised and rectified the data they had on the terrestrial space as they explored it. As a result a new cartography emerged in the Mediterranean region. Towards the end of the 13th century, we find notable centres of map-making in Genoa, Venice and Palma de Mallorca. This cartography was characterised by being aimed at marine navigation and was of a markedly utilitarian nature. The arrival and generalised use of the compass in Europe encouraged the production of nautical charts based on calculations that indicated the position of a ship and its distance from different ports. These maps, focussed on maritime navigation routes, are known as *portolans* or *portolan charts*. They showed the outstanding features of the coast, the visible outline of the coastline, river estuaries, prevailing winds and so on. Many portolan charts were produced in the 14th and 15th centuries.

The greatest work of the Majorcan portolans was the Catalan Atlas *by Abraham Cresques in 1375. The one shown here is a copy from the 19th century.*

In the 16th century, seafaring reached its apogee: in less than a century the discoveries of new lands doubled the area of the known world. Cartographic representations of the Earth were perfected and, for the first time, direct proof of the Earth's sphericity arrived with the circumnavigation of the world by Ferdinand Magellan (1480–1521) and Juan Sebastian del Cano (1476–1526). The dimensions of the terrestrial globe would soon be measured again.

THE FIRST DIRECT PROOF OF THE SPHERICITY OF THE EARTH

The first circumnavigation voyage (1519–1522), which provided direct proof of the Earth's roundness, was initiated by Ferdinand Magellan and concluded by Juan Sebastian del Cano. Magellan captained an expedition that sailed with five ships from Sanlúcar de Barrameda (Cadiz) on 20 September 1519. He crossed the Atlantic and arrived at the Brazilian coast near present-day Rio de Janeiro. He continued to the River Plate and carried on to Patagonia. It was there that Magellan discovered and crossed the strait that bears his name. He crossed the Pacific on a long voyage wracked with difficulties and hardships for the crew. They discovered the island of Guam in the Mariana Archipelago and arrived at the Philippines in March 1521, where Magellan died on 27 April 1521. After his death the expedition was then led by Juan Sebastián del Cano. From the Moluccas Islands he crossed the Indian Ocean, rounded Africa and arrived back at Sanlúcar de Barrameda on 6 September 1522, on a ship named *Victoria*, and thus completed the first circumnavigation of the Earth.

Triangulation networks for measuring meridian arcs

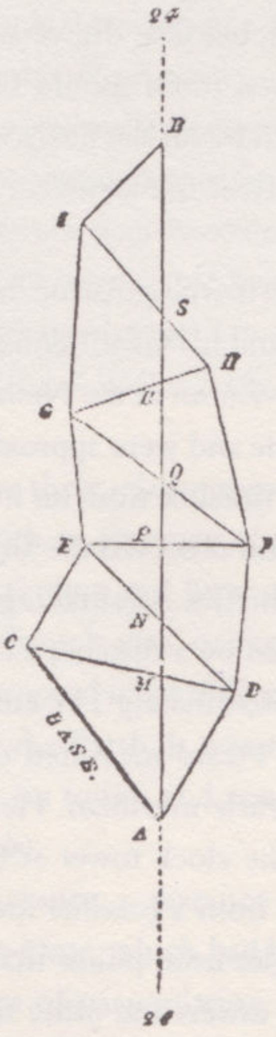
Between 1669 and 1670, French abbot and astronomer Jean Picard became the first person to measure the Earth with a high level of accuracy. He carried out a very extensive operation using the principles of geodesic triangulation. He availed himself of a method used by Willebrord Snellius (1580–1626), an astronomer, mathematician and teacher in Leiden. Snellius, after having planned and carried out measurements in 1615, published his methods in 1617 in his work *Eratosthenes Batavus* (The Dutch Eratosthenes), in which he laid the foundations of geodesics. The method he proposed was to measure the terrestrial circumference by determining the length of a meridian arc calculated by triangulation.

From the point of view of geometry, triangulation is the use of triangles and trigonometric ratios between its angles and sides to calculate the positions of points, measure distances or determine the areas of figures. In the case that we are dealing with here, it is the application of triangulation to the measurement of an arc of meridian.

In his novel *The Adventures of Three Englishmen and Three Russians in Southern Africa*, the brilliant novelist Jules Verne (1828–1905), gives a clear description of the set of operations required in triangulation:

> "To help those readers who are not sufficiently familiar with geometry to understand this geodetic operation known as triangulation, we have borrowed these lines from *Leçons nouvelles de Cosmographie* [*New Lessons in Cosmography*] by Monsieur H. Garcet, teacher of mathematics at the Lycée Henri IV. With the aid of the adjoining diagram – on the next page— this curious task is easy to understand:
>
> "Let *AB* be the arc of meridian in question and whose length we wish to ascertain. We very carefully measure a base *AC*, going from tip *A* of the meridian to the first station *C*. Next, on each side of the meridian we choose the other stations, *D*, *E*, *F*, *G*, *H*, *I*, etc., from each of which we can see the neighbouring stations, and with the theodolite we measure the angles of each of the triangles *ACD*, *CDE*, *EDF*, etc. formed by the stations. This first operation enables these different triangles to be solved, as for the first one, *AC* and the angles are known, and side *CD* can be calculated; for the second one, *CD* and the angles are known, and side *DE* can be calculated; for the third triangle, *DE* and the angles are known, and side *EF* can be calculated, and

so on. Next, at *A* the direction of the meridian is determined by the ordinary method, and measurement is taken of the angle *MAC* that this direction forms with base *AC*; therefore, in the triangle *ACM* we know side *AC* and the adjacent angles, and we can calculate the first stretch *AM* of the meridian. At the same time we calculate angle *M* and side *CM*; so, in triangle *MDN* we know side $DM = CD - CM$ and the adjacent angles, and can calculate the second stretch *MN* of the meridian, the angle *N* and side *DN*. Therefore, in triangle *NEP* we know side $EN = DE - DN$, and the adjacent angles, and can calculate the third stretch *NP* of the meridian, and so on. It can be seen that in this way, piece by piece, the total length of arc *AB* can be determined."



So, first and foremost, to carry out triangulation, the side of a triangle – known as the base – needs to be known as precisely as possible because the accuracy of all the other measurements will depend on it; in practice, this is the most difficult and arduous operation. The base must be as long as possible so as to be able to reduce

errors to a minimum. From each of its tips, measurements are made of the angles it forms with each of the sides that join it to an appropriately chosen third vertex, which determines the first triangle in the network.

If we know two angles and one side (the base) of a triangle, by using trigonometry we can calculate the remaining elements (the third angle and the other two lengths). In this way we get the complete triangle, and any of its three sides can be used as a new base for the second adjacent triangle. If we carry on adding triangles, one beside the other, with our network we will finally join up the two tips of the meridian arc that we want to measure. The astronomical latitudes and longitudes of those points must be determined with the highest possible level of accuracy.

Once the value of the baseline length is known, it should be calculated at its horizontal projection. In general, because the vertices are not necessarily at the same height, the distance between them should be reduced. That is to say, the projection should be considered to be on the horizontal plane or reference surface. Snellius calculated the way to correct the formulae in order to adapt them to the Earth's curvature.

The systematic use of modern triangulation networks stems from the first measurements made by Snellius and his measurement of the distance between the towns of Alkmaar and Bergen-op-Zoom in the Netherlands. These two places were separated by one degree of latitude and were approximately on the same meridian. Snellius used the reference of the distance from his home to the local church tower. He built a chain of 33 triangles and observed the angles of the triangles by using a quadrant measuring two by two metres. After taking the measurements he worked out that the towns were separated by a distance of 117,449 yards (107.395 km). In actual fact, the distance is approximately 111 km.

By using Snellius's methods, Picard measured the distance corresponding to one degree of latitude on the Paris meridian. He used a chain of 13 triangles from Malvoisine, near Paris, to the clock tower at Sourdon, a town near Amiens. The network of triangles started from a baseline measured on the ground that he completed by measuring the angles from points that were visible from each other and situated in watchtowers, bell towers and other similar constructions.

Picard's measurement was the first in which telescopes were used together with quadrants. He developed his own measuring instruments. He had mobile quadrants fitted with binoculars and a micrometer by the French astronomer Adrien Auzout, which provided him with an accuracy of a few seconds of a degree. For accurate measurement it was necessary to find out the differences in altitude of the different

JEAN PICARD (1620–1682)

Educated at the Jesuit College of La Flèche, Jean Picard worked with Pierre Gassendi, a mathematics teacher at the Collège Royale de París (nowadays known as the Collège de France). After Gassendi's death in 1655, Picard became a teacher of astronomy at that same institution and in 1666 became a member of the recently founded Académie Royale des Sciences. He designed a micrometer for measuring the diameters of celestial objects (the Sun, the Moon and the planets) and in 1667 added a new telescopic sight to the quadrant, making it much more efficient for making observations. He made huge improvements to the precision of measurements of the Earth by using the Snellius method of triangulation, and he applied scientific methods to mapmaking. Together with the Danish observer Ole Rømer at the Uraniborg Observatory, in 1671 he observed 140 eclipses of Io, Jupiter's satellite. Using data the two men had gathered, Rømer was able to make the first quantitative measurement of the speed of light.

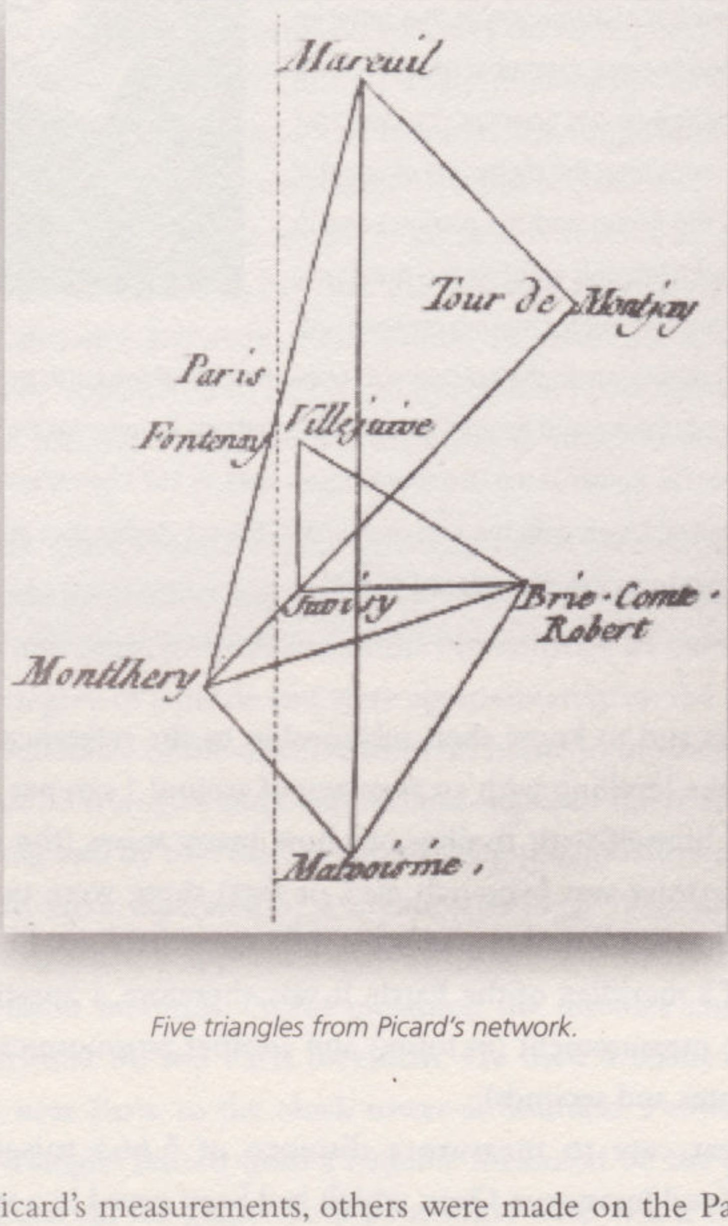
observation points and to know their relationship to the reference plane. He was able to calculate the levelling with an accuracy of around 1 cm per kilometre.

Picard tasked himself with finding out how many toises (the units of length that he used; one toise was 6 French *pied*, or feet) there were in a straight line between Malvoisine and Sourdon, and what difference there was in latitude, on the circumference of a meridian of the Earth. It was, therefore, a question of carrying out one geodesic measurement (in toises) and another astronomical measurement (in degrees, minutes and seconds).

He took great care to measure a distance of 5,663 toises on the road between Villejuif and Juvisy-sur-Orge, which had been paved in a straight line, and he completed the rest by means of triangulation. As a standard measure he used Châtelet's toise, also known as the 'Paris toise' (the length of which would later, at the end of the 18th century, be rated at 1,949 m), and calculated the length of a meridian arc of one degree at 57,060 toises.

Thanks to the improvements made by Picard's instruments and their accuracy, his measurement is considered the first reasonably exact measurement of the Earth's

radius. His measurements gave a result of 110.46 km for one degree of latitude, which corresponds to a terrestrial radius of 6,328.9 km (the radius at the equator is nowadays calculated at 6,378.1 km; the polar radius at 6,356.8 km and the mean radius at 6,371 km). Isaac Newton used Picard's data in his law on universal gravitation.



Five triangles from Picard's network.

Following Picard's measurements, others were made on the Paris meridian by Giovanni Domenico Cassini (1625–1712), the director of the Paris Observatory, and his son Jacques Cassini (1677–1756), who succeeded him in his post. Jacques measured the meridian between Dunkirk and Perpignan, and published the results in 1720. Later, between 1733 and 1740, together with his own son César Cassini, Jacques carried out the first triangulation of France, which in 1745 led to the publication of the first proper map of the country.

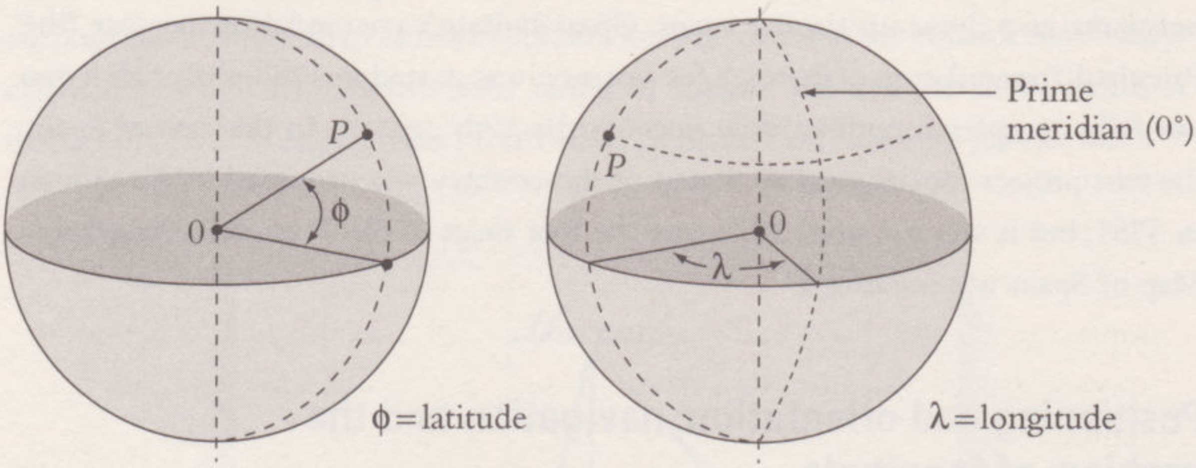
Other countries subsequently carried out measurements with triangulation networks and drew up similar maps. Great Britain's triangulation project (the Principal Triangulation ofBritain), for instance, was started in 1783, although it was not fully completed until halfway through the 19th century. In the case of Spain, the first project for drawing up a map of the country was proposed by Jorge Juan in 1751, but it was not until 1875 that the first sheet of the National Topographic Map of Spain was published.

Positioning and orientation: navigation and the problem of longitude

To determine the position of any point on the plane we can use a system of Cartesian coordinates starting from two perpendicular axes, the abscissa axis (x) and the ordinate axis (y); by giving two values (x,y) we can unequivocally determine a unique point on the plane. Similarly, to precisely position any place on the Earth, i.e. when considered as a spherical surface, it is sufficient to know two values, the place's latitude and longitude (geographic coordinates). They now take the role of axes on a great circle passing through the poles, in other words, the meridian that we have chosen to use as a baseline or origin of our measurements (meridian 0°), and the great circle corresponding to the terrestrial equator.

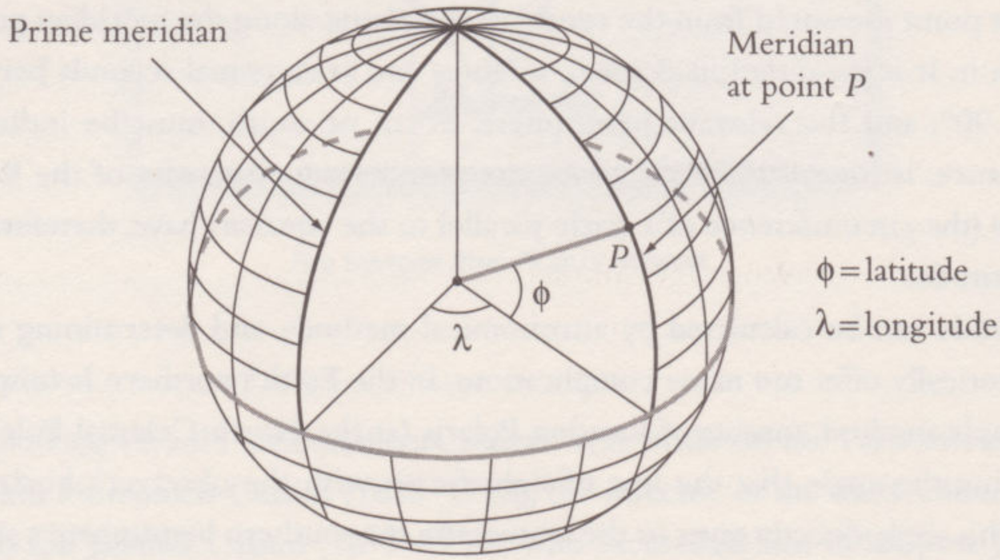
The latitude of a point on the Earth is the angular distance between the equator and that point measured from the centre of the planet along the meridian passing through it. It is measured in degrees, minutes and sexagesimal seconds between 0° and 90°, and the relevant hemisphere, north or south, must be indicated; for instance: latitude 41° 24' 14" N. All the points located on one of the Earth's parallels (the circumference of a circle parallel to the equator) have, therefore, the same latitude.

Latitude can be calculated by astronomical methods, and determining it did not historically offer too many complications. In the Earth's northern hemisphere, one simple method consists of locating Polaris (at the North Celestial Pole) and measuring the angle that the line of sight forms with the observer's horizontal plane. This angle directly gives us the latitude (in the southern hemisphere a similar system would be used by taking the Southern Cross as a reference). There are also other methods for determining latitude during the day, for example, by measuring the height of the Sun over the horizon and using tables with the coordinates of its motions along the ecliptic throughout the year.



Latitude and longitude of a point P *on the sphere.*

Longitude is the value of the angle between the meridian baseline (in fact, a semi-meridian), which is taken as the origin (0°), and the meridian that passes through the point, measured on the equator from the centre of the Earth. It is also measured in degrees, minutes and sexagesimal seconds, as is latitude. The measurement is taken between 0° and 180°, in this case indicating east or west according to the direction taken from the base meridian. For instance, longitude 2° 14' 50 " E. All points situated on a semi-meridian between the Earth's two poles therefore have the same longitude.



Latitude and longitude are calculated from the equator and from a meridian that is taken as a reference (the prime meridian or meridian of longitude 0°).

Nowadays, the Greenwich meridian is the one normally taken as the prime meridian. Before it, however, many other meridians served as the reference meridian. Calculating latitude on a ship, as we described above, is simple and the problem was easily solved. As for longitude, if references on land are within sight then calculating it is relatively easy, but on the high seas it causes serious problems.

The problem of the calculation of longitude began to acquire vital importance after the discovery of America by Christopher Columbus. Longitude was calculated in an approximate fashion by deducing the distance that the vessel had covered from east to west or vice versa. To do this, sailors used the chip-log, an instrument used by them to measure a ship's speed. The chip-log carried a rope wound into a spool and it could be held in such a way that it could rotate freely; the rope had a series of equidistantly spaced knots. A sailor would toss the chip-log over the stern and let out the first part so that it could stabilise in the water. When he felt the first knot in his hand he would shout "Mark!"; another sailor would then begin to measure the time with a sand clock; when the man with the sand clock saw that all the sand had passed through he would inform the first sailor, who would then halt the movement of the rope and announce the number of knots that had passed, for instance, "three knots and a half" or "six knots and a quarter". That is the origin of why even today the speed of ships is measured in knots.

Of course, such a rudimentary method for calculating longitude led to great inaccuracies and some real disasters in seafaring, so much so that the calculation of longitude became a strategic priority for countries with interests overseas. In the 17th century and at the beginning of the the 18th, calculating longitude was a primary technological problem to be solved.

In theory, the calculation of longitude can be reduced to measuring the difference in time between a reference point (the departure port or meridian zero) and the position of the vessel. When the Sun passes over the meridian where the observer (the ship) is located, if the exact time at the departure point is known, it is possible to discover the longitude, that is, the distance in degrees to the departure port and, therefore, also to meridian zero. This is due to the fact that the difference in time between the two meridians can be converted into a difference in degrees. As the Earth takes 24 hours to rotate 360°, in one hour it covers one 24th of its rotation, that is, 15°. If in one hour, or what comes to the same thing, in 60 minutes, it rotates 15°, in general terms a difference of 4 minutes stands for one degree of longitude.

Longitude can therefore be calculated by using the differences in the time on noon (when the Sun crosses the meridian) between two places, and that, in principle,

could be done by carrying out astronomical observations and measurements. It was thought that this could perhaps be done by observing eclipses, but on the high seas this is not a feasible technique and, what's more, eclipses were few and far between.

CALCULATING LONGITUDE BY OBSERVING ECLIPSES

Let's suppose that we know the time that an eclipse is going to occur at a determined place (on land, at an astronomical observatory, etc.) and that we are on the high seas. If we can measure, at our local time, when that eclipse occurs, we shall be able to calculate the longitude. To use this method only requires having tables that give information on the times that the eclipse will occur at the reference point (and mathematics, of course!). In the 16th century the observation of eclipses could be useful as a tool on land, but was not a useful technique for determining longitude at sea as it was difficult to keep the instruments in a fixed place due to the motion of the sea and, above all, because eclipses are not frequent occurrences. Every year, in fact, there could be from two to a maximum of five eclipses of the Sun, and if we count both solar and lunar eclipses, there would be a minimum of two eclipses and a maximum of seven, the average being four. In the 20th century, 375 eclipses were counted: 228 of the Sun and 147 of the Moon. Apart from the scarcity of eclipses, another problem is whether they are visible or not, as visibility will be subject to a determined place's meteorological conditions, not least whether or not it is cloudy.

The low frequency of this predictable astronomical phenomenon was improved thanks to Galileo's discovery of Jupiter's satellites in 1610. When they orbit the planet, Jupiter's moons are eclipsed and they appear and disappear. Those eclipses take place around one thousand times a year and it is possible to predict when they will take place. Although it was an adequate method, and it could be used to determine longitude on land, at sea difficulties arose on account of ships' instability and because the observations had to be done at night, only in part of the year and, furthermore, with a clear sky.

The problem of longitude at sea continued to be a problem. Measuring the position of the Sun was good for determining the local time. However, how could the reference time be known without having sufficiently accurate clocks? Among other factors, the working of pendulum clocks was disturbed by a ship's swaying, and the variations in latitude could cause the clocks to run fast or slow. A ship's clocks invariably failed to keep the same time as those at the departure port, giving rise to considerable errors in calculating longitude.

In 1714, the British Parliament offered a prize of £20,000 to whoever was able to provide a method or instrument to solve the problem of determining longitude on the high seas. The main winner was English watchmaker John Harrison (1693-1776), who after decades of work managed to build a very accurate ship's clock, or chronometer, that was tested on a ship bound for Jamaica in 1761. It was running for 147 days until it returned to England and showed a variation of only 1 minute and 54 seconds.

The problem of longitude could therefore be considered solved – although there was a real economic obstacle: the price of a Harrison clock was a third as much as an entire ship. Nowadays, the exact position of a ship can be determined thanks to GPS, which we shall look at in Chapter 6.

A non-spherical Earth

Measurements of the Earth, including those of Picard, were taken by supposing that the Earth was completely spherical. However, just a few years after Picard's measurements, between 1671 and 1673, the French astronomer Jean Richer (1630-1696), one of Giovanni Domenico Cassini's assistants, travelled to Cayenne in French Guiana, where he made a crucial discovery. He observed a change in the strength of the terrestrial gravitational force when he noted that a pendulum's swing was slower in Cayenne than in Paris. He deduced, correctly, that this was due to Cayenne's being further from the centre of the Earth than Paris. News of this discovery reached Europe and shocked the members of the Académie des Sciences. On his return to Europe, Richer set to work to determine the length of the pendulum that in Paris would mark out seconds, in other words, that took one second to go from one extreme point to the other. Similar measurements were also taken in other parts of the world, and it was found that the length varied with latitude. In the light of the existing theories, everything indicated that if the force with which the Earth attracted the pendulum varied in different places, then the shape of the Earth could not be completely spherical.

Taking into account Richer's work, in 1687 Newton published his famous *Mathematical Principles of Natural Philosophy*, in which he set down the foundations of mechanics. He provided a mathematical explanation for the shape of the Earth which he linked to his brilliant and innovating theory on universal gravitation. If our planet was considered as a homogeneous, fluid mass which is subject to rotation, it had to be squashed at the poles. He announced that the flattening was 1/230, that

is to say, supposing that the cross section of the Earth were an ellipse, the greatest axis would exceed the smallest by a 230th part of it.

In France, Jacques Cassini in 1720 published the work *De la grandeur et de la figure de la Terre*, in which he rejected the flattening of the poles maintained by Newton, and backed up his assertions by presenting astronomical and geodetic observations based on his own measurements of the meridian Collioure-Paris-Dunkirk (though some members of the Académie believed they could see inaccuracies in them). Cassini accused Newton of being speculative, and maintained that the Earth was an ellipsoid flattened at the equator. So, was the Earth in the shape of a lemon or an orange? A controversy blew up over the Earth's shape which came to involve London's Royal Society and Paris's Académie des Sciences in a debate imbued with nationalist sentiment that set French and English science against each other (again).

With the aim of putting an end to the polemic, the Paris Académie des Sciences decided to carry out accurate measurement of the arc corresponding to one degree of central angle on a meridian at latitudes that were as distant from each other as possible. For this purpose two expeditions of scientists – astronomers, mathematicians, naturalists and others – were organised: one to Lapland under the direction of Pierre Louis Moreau de Maupertuis (1698–1759), which included Le Monnier, Clairaut, Camus, the Swede Anders Celsius and Abbot Outhier. The other set out to the Viceroyship of Peru (specifically to modern Ecuador) headed by the astronomer Louis Godin (1704-1760), with geographer La Condamine, the astronomer and hydrography specialist Bouguer and botanist Jussieu. The South-American scientist Pedro Vicente Maldonado joined them at Guayaquil. Other participants were the clockmaker Hugot, the engineer and sketcher Morainville, the ship's captain Couplet, the surgeon and botanist Séniergues, the technician Godin des Odonnais (Godin's nephew) and the cartographer and naval engineer Verguin.

At that time the Viceroyship of Peru was Spanish territory, and permission had to be sought from the Spanish Crown to take the expedition into those territories in the equatorial Andes. In return for granting this permission, two brilliant young Spanish officers from the Cadiz Naval Academy, Jorge Juan and Antonio de Ulloa, joined the expedition.

The expedition to Lapland (1736–1737) was equipped with the capacity for calculation in the shrewd form of mathematician Clairaut, and took a relatively short time to produce results. They were also able to count on the help of the Swedish army for setting up the stations. The scientists carried out the triangulation on the long days of the northern summer in a chain along 100 km between Kittis and Torneå.

They took the astronomical measurements in spring and autumn when the nights are long but not excessively cold. The baseline was measured by taking advantage of the frozen surface of a river. The final result of Maupertuis's measurements for the degree of meridian was 57,438 toises at the mean latitude of 66° 20'. By comparing it with the 57,060 toises that Picard had measured near Paris at an approximate latitude of 48°, it could be now be confirmed that the Earth was a spheroid (orange) that was flattened at the poles.

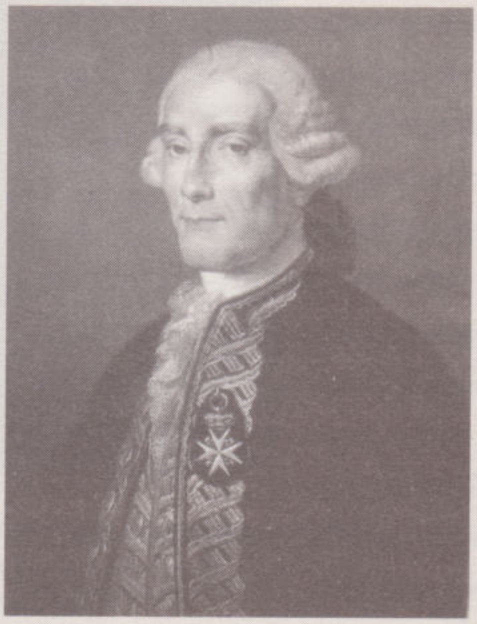
A sketch of goniometer measurements being taken for carrying out triangulation. A illustration from Jules Verne's novel The Adventures of Three Englishmen and Three Russians in Southern Africa.

On the other hand, the expedition to America turned into an saga that lasted a decade. It sailed from La Rochelle in the spring of 1735 and arrived at Quito a year later. The members of the expedition came up against all kinds of problems, which were added to the constant disputes between the French scientists added to by the harsh climate; difficult, rugged terrain; numerous financial problems, and not least the expedition splitting into two in 1741. The measurements and triangulation

were particularly difficult processes due to the complications brought about by the mountainous Andean territory and the altitude of more than 4,000 m that they had to climb to. They decided to carry out a large-scale triangulation of 43 triangles over a length of 354 km so as to measure not just 1 degree of meridian but 3. The results for the degree of meridian were 56,763 toises according to Bouguer (1749) and 56,768 toises according to Juan and Ulloa (1748) and to La Condamine (1751). To use the simile of the lemon or the orange, the Earth was more an orange than a lemon (or as Voltaire formulated it a watermelon not a cantaloupe). The measurements and mathematical calculations finally seemed to prove Newton right.

JORGE JUAN AND THE SAN FERNANDO ROYAL OBSERVATORY (CADIZ)

The Spanish naval officer Jorge Juan y Santacilia (1713–1773), a member of the expedition that set out to measure the length of a meridian degree at the equator, was one of the scientists who most contributed to the modernisation of Spanish science in the 18th century. His legacy still endures as, among other things, he was responsible for founding the Royal Naval Observatory (ROA) in 1757 at San Fernando in Cadiz. In addition to being an astronomy and geodetic observatory, the present-day Royal Naval Institute and Observatory is also a centre for scientific research and cultural diffusion. Among its other activities, it deals with the calculation of ephemerides, the publication of the *Nautical Almanac*, meteorological, seismic and magnetic observations, and the scientific determination of time. It is tasked with calculating and publishing the official Spanish time (Universal Coordinated Time: UTC) and is also the holder of the official Spanish metric measurements.



Jorge Juan y Santacilia. Naval Museum, Madrid.

Chapter 5

Measuring the Metre

In this chapter we shall make a brief journey through the history of the metre. Starting off from the measurements used in the 18th century, we shall look at the difficulties caused by using a multitude of different units, and the historical circumstances that led to the decision to take on the task of establishing a universal system. We shall deal with the conditions required of the new unit of length, the different proposals put forward, and why it was decided to measure a meridian arc in order to set the new standard. We shall look at the mathematical conditions of the task (triangulation), the instruments used (the Borda circle) and those who were the protagonists (Jean-Baptiste Delambre and Pierre Méchain), as well as some issues concerning the 'adventures and misadventures' of the geodetic expeditions. Finally, we shall look at how the metric system came to be accepted, the reluctance to accept it in certain countries and some of the conflicts that it has generated.

The need for a universal measure

In the 18th century it was impossible to buy milk in litres or potatoes in kilos; such units did not exist. The same weights and volumes of goods were measured in different units depending on the country, the region, the county or even the town; the official standards of measure for some places were displayed at the city gates.

The vara or yard in Valencia and Castellón was approximately 0.906 m, while the one in Teruel measured 0.768 m. By buying cloth at so much per yard in Castellon and selling it for the same price in Teruel one could make a profit of 18%. The league was another unit that varied from region to region – from the common league in Spain, measuring 5,572 m, to the French league, of 3,898 m. The variability of the foot was also notorious, from the foot of Burgos (0.278 m) to the long French foot (0.324 m). The foot is a particularly prominent unit and is closely related to the railway system. The present-day international railway gauge, which originated in Britain, is four Imperial feet, eight and a half inches (1.43 m). The pound weight also had numerous versions: before the metric system

(officially the *Système métrique décimal; SDM*) was established, there were 391 different pounds in use in Europe.

WHAT A LOT OF FEET!

Below is an example of the variety of different measurements that can correspond to the same unit; in this case, feet as an anthropomorphic measurement that appeared in many different countries. The equivalents are given in metres:

Burgos foot	0.278 m
French foot	0.324 m
Rhineland foot	0.314 m
Roman foot	0.297 m
Amsterdam foot	0.283 m
Swiss foot	0.300 m
English foot	0.304 m
Russian foot	0.305 m
Egyptian foot	0.225 m
Austrian foot	0.316 m

The existence of numerous measures and their variability depending on the region made trade more complicated and caused enormous difficulties for the traffic of goods. The lack of a common definition enabled units of measure to become an instrument of domination, and unifying them was one of the objectives of the French Revolution. With this aim in mind, on 9 February 1790, Claude-Antoine Prieur-Duvernois (1763–1832), nicknamed 'Prieur de la Côte-d'Or', a military engineer in charge of the requisition of arms and munitions for the revolutionary forces, petitioned the French National Assembly.

In addition to the social progress implied by this task, the scientific community was also concerned about the need to establish universal measures, and the Paris Académie des Sciences played an important role in the process. Units and standards were expected to serve for measuring the different magnitudes required

by the developing discipline of physics. They were required to measures general characteristics common to objects that were seemingly different, such as, for instance, oil and wine. If they were both to be measured taking into account their common property – their liquidity – there was no sense in using different units, one for oil and a different one for wine; a common unit and standard was needed. Years later, this task would be accomplished by the litre.

One of the basic magnitudes is length, a characteristic common to numerous objects. There are innumerable different standards for measuring length, and consequently it seemed logical to start by defining a standard unit of length. Furthermore, if a standard was to be established, it was important to have precise instruments both for measuring the object that would become the standard and for reproducing it and making as many copies as necessary. Conditions were just right for carrying out the task: there had been great technological progress enabling the manufacture of ever-more-precise measuring tools.

The ideas on equality proclaimed by the French Revolution had a decisive influence on the conditions that the new universal units had to meet. There were three of these conditions: they would be accepted in all countries; they would be unchangeable; and they must not be anthropomorphic (as were the foot and the palm, for example).

The decision to choose the meridian

What conditions was the new measurement expected to fulfill? What proposals were considered? How and by whom was the final decision taken? And, lastly, why was it decided to measure a meridian arc and which one was chosen? These questions determined the three expeditions that were made to establish the metre as a standard for measuring length.

Stipulations for the new measure

According to the instructions from the Académie des Sciences, it was necessary to find a unit of length that was based on nature, as such a unit would be considered unalterable and would belong to all humanity. In this way the new unit would be stable, and would stand the test of time. But what was the natural phenomenon that could serve as the basis for the new unit of measure and would meet all those requirements mentioned above? There were three alternatives for the standard unit:

the first was the length of a pendulum; the second was the size of an arc of the equator; and the third was the size of a meridian arc.

What approximate length should the new unit of measure be for it to be useful in daily life? It was thought that the length of half a toise could be appropriate as a starting point. A start could now be made on analysing the three proposals, which was done by a commission from the Académie des Sciences at the request of the National Assembly.

Three proposals

Three months after Prieur de la Côte-d'Or requested implantation of a single universal standard, the Assembly met on 8 May 1790 to debate the different possibilities.

DECIMAL AND NON-DECIMAL UNITS

The superior efficiency of a decimal system as compared to a non-decimal system becomes evident when calculations are being made. For example, to calculate the sum of two lengths in the decimal system, say, 3 m + 7 cm = 3.07 m or 307 cm, it is enough to make a change in the equivalences which go in multiples of ten, hence the name "decimal".. In contrast, to add 1h 35 minutes + 42 minutes we cannot write 1h 77 minutes, but instead have to write 2 h 17 minutes. That is because the relationship between an hour and a minute is in multiples of 60 and not of 10.

A system that is non-decimal also entails more difficulties when operating with fractions of the unit. Thus, $\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$ in either of the two situations, but in the case of the decimal system,

$$\frac{3}{4}\text{of a metre} = 0.75\text{ m}\left(\frac{3}{4}=0.75=\frac{75}{100}\right)$$

or also 75 cm, while $\frac{3}{4}$ of an hour = 45 min, and cannot be expressed as 75 min, or as 7.5, nor with any other expression that reminds us of the fractions of the denominator, 10, 100, etc., as in this latter case, $\frac{3}{4}=\frac{45}{60}$.

While the equivalents between hours, minutes and seconds are not in the decimal system, the fact that they are a sexagesimal system means that they are relatively simple to use, and for that reason the system has survived for many centuries right up to the present day. The issue was much more complicated with other non-decimal equivalencies which fell out of use when the metre was introduced.

In the session there were two particularly relevant proposals, one in support of the metric reform and another on the length of the pendulum as the basis for the new unit. Charles Maurice de Talleyrand, the President of the Assembly and the Bishop of Autun, proposed that the standard unit should be the length of a pendulum the swing of which took one second at a latitude of 45°. The second proposal, put forward by Prieur de la Côte-d'Or himself, was that the new system should also be decimal. It was to divide the length of the pendulum into three equal parts. Each part was a foot, and from that a decimal system was to be built: the foot had ten inches, and an inch was to have ten lines.

Once all the speeches had been heard, the Assembly asked the Académie des Sciences to draw up a study and analysis of the different proposals so as to decide how the reform of the system of measures should be implemented. The Académie appointed a commission comprising the most illustrious scientists of the time: Pierre-Simon Laplace, Joseph-Louis de Lagrange, Jean Charles de Borda, Gaspard Monge and Nicolas de Condorcet among them. On 19 March 1791, the commission issued a report containing three alternatives for choosing a unit of measure that would be acceptable for all time and all peoples.

- The length of a pendulum whose half-period was one second at a latitude of 45°.
- A quarter of the equator.
- A quarter of a meridian.

The final decision

The Académie des Sciences was in favour of a decimal system, as Prieur had proposed, and of the three alternatives in the report, it favoured establishing the new measure based on the length of the meridian. Once the Académie's report had been issued, the final decision rested with the Assembly. On 26 March 1791 the Assembly ratified the Académie's decision and chose the third option: "The fourth part of a terrestrial meridian shall be the true unit of measure, and the ten millionth part of that length shall be the common unit." In the same session it was decided to adopt the name *mètre* (from the Greek μέτρον, *metron*, meaning "measure") for the unit.

What were the reasons that prompted the Académie des Sciences to opt for the measure of the meridian? There were already some good calculations of the

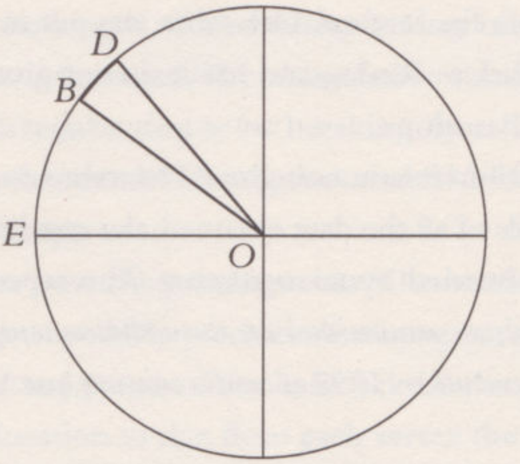
length of the meridian, but it would be necessary to carry out a much more ambitious operation of measurements to reach the level of precision required by the definition of the new unit. Why start up a project that would surely cost much more than that needed for measuring a pendulum? The commission's decision caused anger among some people, among them Jean-Paul Marat, who had already seen the Académie des Sciences reject his scientific work (most of it on physics) on several occasions. The Commission rejected the measurement of the equator on account of the difficulties in carrying out measurements in regions which had been insufficiently explored.

In this respect, the experiences of the previous measurement project in the Viceroyship of Peru were no doubt a dissuading factor. But why was the length of the pendulum rejected? It would have been a relatively easy and cheap option. The Commission argued that the unit of length could not depend on the unit of time and the value of gravity. Is time less elemental than length? These are contradictions of history, as subsequent definitions of a metre that were carried out to adjust the measure to the needs of scientific and technological development established a link between both magnitudes, as we shall see later.

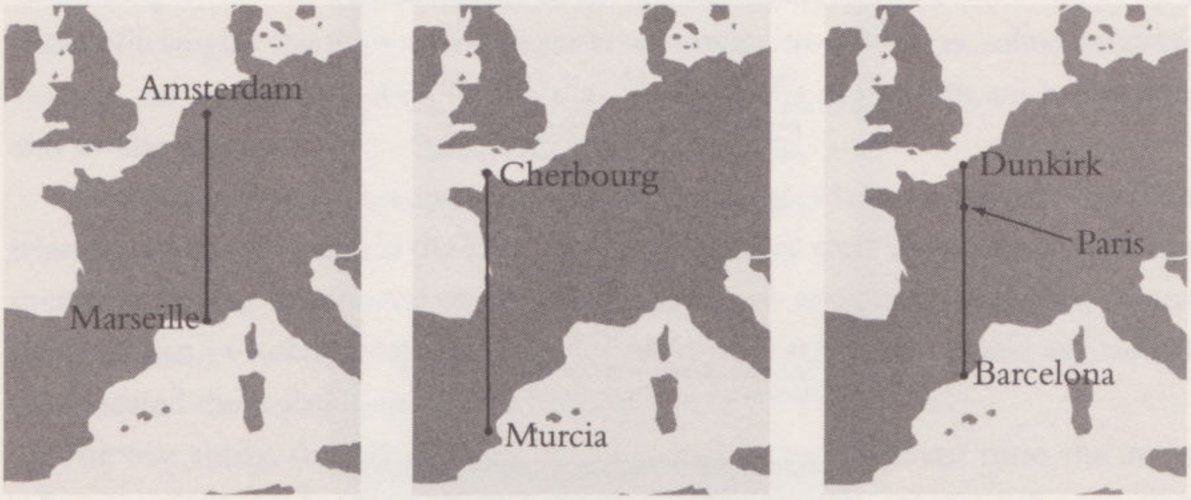
So, what was the reason for choosing the measurement of the meridian? The answer is not at all clear and there are different suppositions that can be made. Some historians attribute the decision to the fact that one of the Commission members, Borda, was the inventor of a very advanced instrument for measuring angles. The measuring of the meridian would serve to show how excellent his instrument was and would definitely establish it as a tool for making topographical and astronomical calculations.

Which meridian arc?

As it was not possible for cartographers to measure all of the quadrant from the North Pole to the equator, they were to measure a meridian arc that would be as long as possible, and then they would extrapolate the results to the whole quadrant. For terrestrial deformation or flattening to have the least possible effect on the measurements, it was thought best to choose an arc around latitude 45° with its end points at sea-level and no large mountains in between. The two greatest European mountain chains, the Alps and the Carpathians, would therefore have to be avoided. Three possible arcs were suggested: Amsterdam-Marseille, Cherbourg-Murcia and Dunkirk-Barcelona.



A diagram showing the quadrant of the meridian used for the definition of the metre. E *is the equator;* B, *the city of Barcelona, and* D, *Dunkirk.*

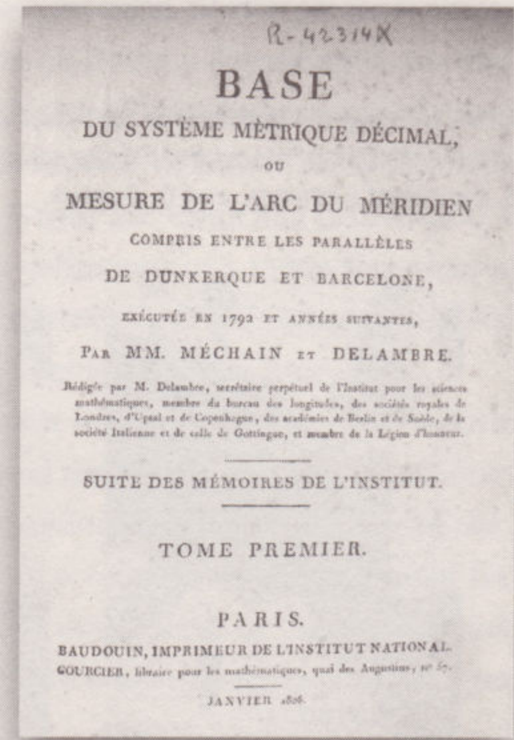


The third option was finally chosen because several measurements of it had previously been made, such as Dunkirk-Perpignan in 1739. The fact that it passed through Paris also probably had an influence on the decision, but it was precisely this issue that prompted the British to withdraw from the project in 1791, though they had initially been willing to take part.

In April 1791, the Commission of the Académie des Sciences entrusted the task to Jean-Baptiste Delambre, Giovanni Domenico Cassini, Adrien-Marie Legendre and Pierre Méchain. Cassini was loyal to the old king and refused to serve under a revolutionary government that had arrested Louis XVI. On 15 February 1792 Delambre was unanimously elected a member of the Académie des Sciences. In May

1792, following Cassini's final refusal, Delambre was put in charge of the northern expedition, from Dunkirk to Rodez, and Méchain was given the expedition to the south, from Rodez to Barcelona.

In January 1806, with Méchain now dead, Delambre concluded a three-volume report containing details of all the data obtained, the conditions of the observations and the calculations obtained by triangulation. The report was entitled: *Base du système métrique décimal, ou mesure de l'arc du méridien compris entre les parallèles de Dunkerque et Barcelone, executé en 1792 et années suivants, par MM. Méchain et Delambre.*



R-42314X

BASE
DU SYSTÈME MÉTRIQUE DÉCIMAL,
OU
MESURE DE L'ARC DU MÉRIDIEN
COMPRIS ENTRE LES PARALLÈLES
DE DUNKERQUE ET BARCELONE,
EXÉCUTÉE EN 1792 ET ANNÉES SUIVANTES,
PAR MM. MÉCHAIN ET DELAMBRE.

Rédigée par M. Delambre, secrétaire perpétuel de l'Institut pour les sciences mathématiques, membre du bureau des longitudes, des sociétés royales de Londres, d'Upsal et de Copenhague, des académies de Berlin et de Suède, de la société Italienne et de celle de Gottingue, et membre de la Légion d'honneur.

SUITE DES MÉMOIRES DE L'INSTITUT.

TOME PREMIER.

PARIS.
BAUDOUIN, IMPRIMEUR DE L'INSTITUT NATIONAL.
COURCIER, libraire pour les mathématiques, quai des Augustins, nº 57.

JANVIER 1806.

The cover of Méchain and Delambre's report.

Triangles: the measurement's mathematical basis

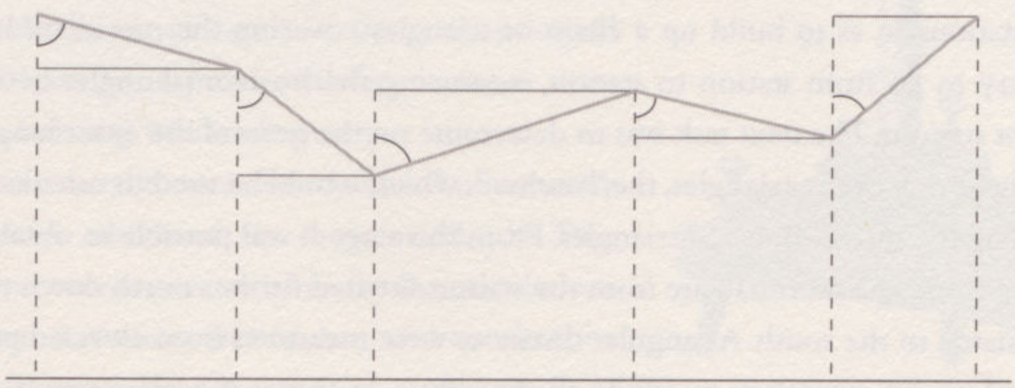
Measuring a meridian arc on the ground was not an easy task two centuries ago. The measurements were carried out indirectly and section by section as the task could not be completed by making one single measurement. It was a case of building up a network of adjacent triangles that stretched along and covered all the area being measured. Once the gridwork of imaginary triangles had been completed, it was enough to find one single measurement of length and the measurements of two of the angles of each triangle so as to work out the calculations of all the sides of the triangles in the network. Finally, by using the drawing of the network

and more calculations, the length of the corresponding arc of meridian could be deduced. This procedure is known as the *triangulation technique*, and it is also used for measuring areas of irregular shapes by breaking them up into triangles. This technique was shown in the section 'Triangulation networks for measuring meridian arcs' in the previous chapter.

Let's remind ourselves how the process works. The baseline is measured on the ground with the maximum possible precision. An imaginary triangle is established with two vertices on the end points of the base and the third one at a point that is normally at an elevated location so that from each vertex the other two can be seen and, therefore, the triangle's three angles can be measured. Once the length of one side and two angles of the triangle are known, the other two sides can be calculated, and these sides can be used as a base for another two triangles. From these lengths calculated during the previous stage, by measuring the angles of the new triangles, their sides can be calculated, and then, in turn, these sides can be used as a base for calculating more triangles. By proceeding in this fashion it is possible to build up a chain of triangles, the sides of which are known and whose vertices, called geodetic vertices, are usually placed on high places, that is, at the top of hills, on bell towers and similar places.

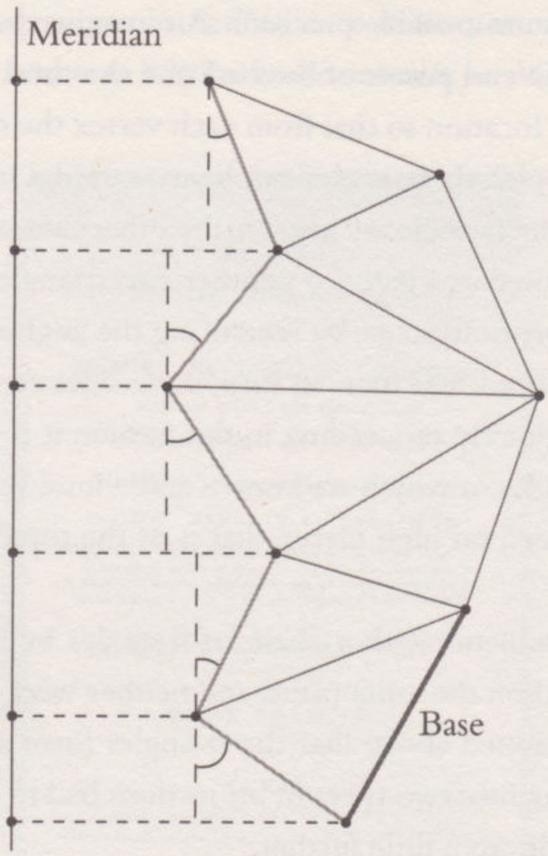
There were two problems with a chain of triangles built up in this way. The triangles are not located on the same plane, and neither were they aligned with the meridian. (It was mentioned above that the triangles form a network that covers the meridian.) Consequently, two types of projections had to be carried out, which complicated the calculations a little further.

For one thing, the sides of the triangles had to be projected onto the same reference plane. To do so, use was made of the zenith angle, which is the one that forms the vertical at a point with the side of the triangle which is to be projected:



Projection of the sides of the triangles on the same horizontal plane.

At the same time, some of the sides had to be projected in the direction of the meridian so that, jointly, they would cover it. To do this, use was made of what is known as the azimuthal angle, which is the angle between the direction of the meridian and the side of the triangle required to be projected (see the second figure overleaf).



Projection of the sides that are to cover the meridian.

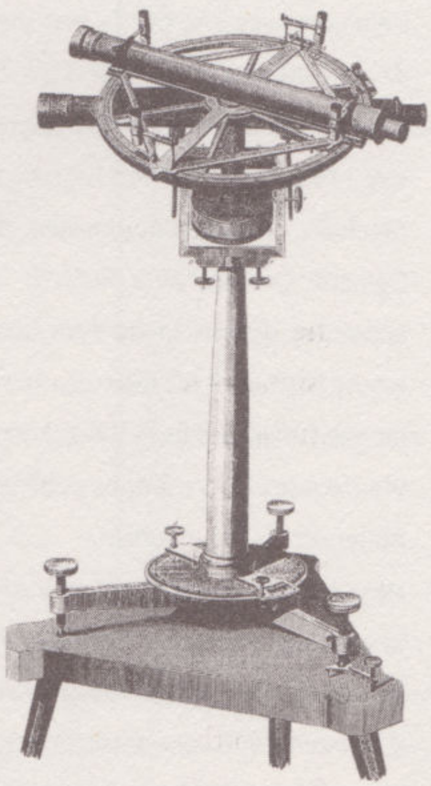
On the ground, the task was to identify the observation stations, i.e. the nodes or geodesic vertices of the triangles, which needed to be visible from at least three other stations so as to build up a chain of triangles covering the meridian. It was necessary to go from station to station, measuring the horizontal angles between adjacent stations. The next task was to determine on the ground the exact length of one side of one of the triangles, the 'baseline', which would be used to calculate the sides of all the interconnected triangles. From this stage it was possible to obtain the distance along the meridian arc from the station situated further north down to the one situated to the south. As angular distances were measured from elevated points, it was of course necessary to adjust all the values to those of a common triangle situated at surface level, such as can be seen in the figure opposite.

The instruments, and the precision of the measurements

For the triangulation technique to be effective, the baseline (the initial length from which all the following ones in the chain would be measured) and the angles of the various adjacent triangles had to be measured with the highest possible level of accuracy. In order to get the best measurements and at the same time prevent the propagation of errors in successive calculations, it was vital to be equipped with precise instruments and to make sure there was adequate visibility between the triangulation's vertices.

For measuring the base, four rulers graded in toises were used, each two toises long. To give an idea of the equivalent size in metres, the unit they were to establish, one toise equals 1.949 m. Each ruler was made of a layer of platinum and another of copper so as to counter the effects of expansion and contraction of the materials due to changes in temperature. To maximise the accuracy of the reading, a slide gauge was used and read with a magnifying glass. The expeditions were also equipped with devices for attaching rulers in a line, aligning and levelling them.

Jean-Charles de Borda (1733–1799), a navy officer and experimental physicist, had invented an optical device used for measuring angles that became known as the Borda repeating circle. The instrument had been constructed by Étienne Lenoir, at the time considered to be the best manufacturer of scientific devices in France. The Borda circle enabled numerous readings of the same angle to be taken without any need for the instrument to be moved. For measuring angles between geodesic vertices the aim was always to try and do it with the best possible visibility, and success depended to a large extent on the meteorological conditions. To improve visibility, in some cases lamps burning whale oil were put to use.


Borda's repeating circle.

It is natural to wonder what level of accuracy could be obtained by these methods. Two baselines were measured: one for the calculations between Lieusaint and Melun, and another for those between Vernet and Salses, in the south of France close to the beach. The Melun baseline

measured 6,075.90 toises (11.8 km) while the Vernet one worked out at 6,006.249 toises (11.7 km). Calculation of the Vernat baseline from Melun, following a chain of 53 triangles along 640 km, gave the result of 6,006.089 toises – an error of just 0.16 toises, that is, 30 cm.

The Dunkirk–Barcelona arc-measuring adventure

The first expedition

The first and best-known expedition (1792-1799) was put in the hands of the astronomers Jean-Baptiste Delambre and Pierre Méchain, both of them followers of Joseph Jérôme Lalande. Delambre and Méchain started off on their travels in June 1792. As his assistants, Delambre, who was in charge of the northern sector of the meridian, took Michel Lefrançais (Lalande's nephew and an apprentice astronomer), Benjamin Bellet (a constructor of instruments and apprentice of Étienne Lenoir) and a servant named Michael.

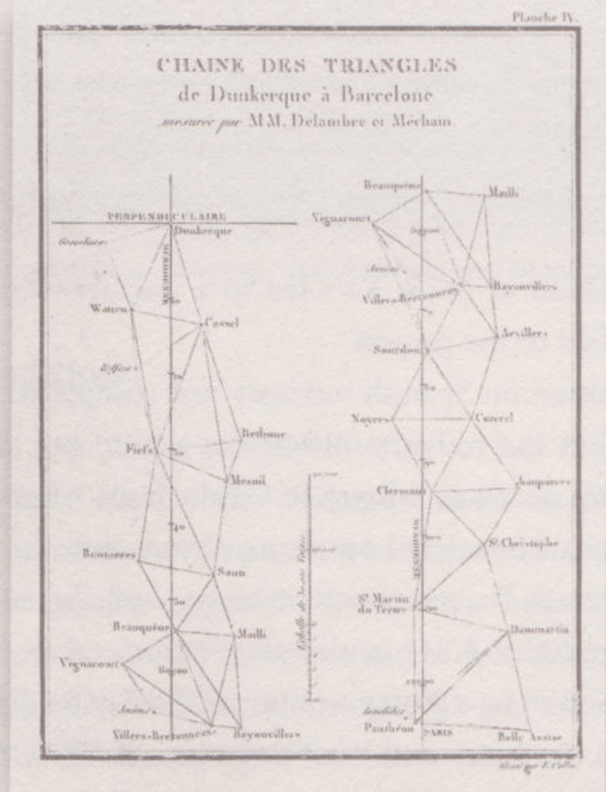
JEAN-BAPTISTE JOSEPH DELAMBRE (1749–1822)

Delambre was one of the first members of the Metre Commission created on the seventh day of Messidor in the year 3 (i.e. in the revolutionary calendar; in the present-day calendar it would be 25 July 1795), together with Lagrange, Laplace, Lalande, Méchain, Cassini, Bougainville, Borda, Buache and Caroché. On Méchain's death in 1804 Delambre was appointed director of the Paris Observatory, a post he would hold until his own death in 1822. In 1809 he received an award from the Académie des Sciences for the decade's best scientific publication in reference to his report on the meridian.

He published several books on the history of astronomy in which he showed a particular interest in the analysis of mathematical calculations: *History of Astronomy* (1817); *History of Astronomy in the Middle Ages* (1819); *A Modern History of Astronomy*, in 6 volumes (1821). The last volume, *A History of Astronomy in the 18th Century*, was published after his death by his disciple Claude Mathieu.

Méchain, a very meticulous astronomer, took charge of the southern section of the meridian (Carcassonne in the Pyrenees region to Camprodon, Puigsacalm, Matagalls and Barcelona). He was accompanied by his assistants, Jean Joseph Tranchot (a military engineer and cartographer), Esteveny (an instrument maker who trained with Lenoir) and a servant named Lebrun. At the beginning of July, 1792, they arrived in the Pyrenees region and by October they were in Barcelona. The report's first volume, which was written some years afterwards by Delambre, contains the whole triangulation chain.

On top of the problems to be expected of such a large geodesic operation, there were others of a diverse nature. On May 1 1793, Méchain went on a trip with the Catalan scientist Francesc Salvà to visit a pumping station on the outskirts of Barcelona. An unfortunate accident caused the pump's almost two and a half metre long lever to strike Méchain in the chest and fling him against a wall. He fell to the floor, apparently dead. That night, Doctor Francesc Santponç, "the best surgeon in the city", was summoned. The right-hand side of Méchain chest was caved in, the ribs smashed and his collar-bone broken in several places. Nevertheless, he recovered very slowly, although six months later his arm still hung lifeless.



A fragment from the Dunkirk to Rodez triangulation, including Paris, from Méchain and Delambre's report.

PIERRE FRANÇOIS ANDRÉ MÉCHAIN (1744–1804)

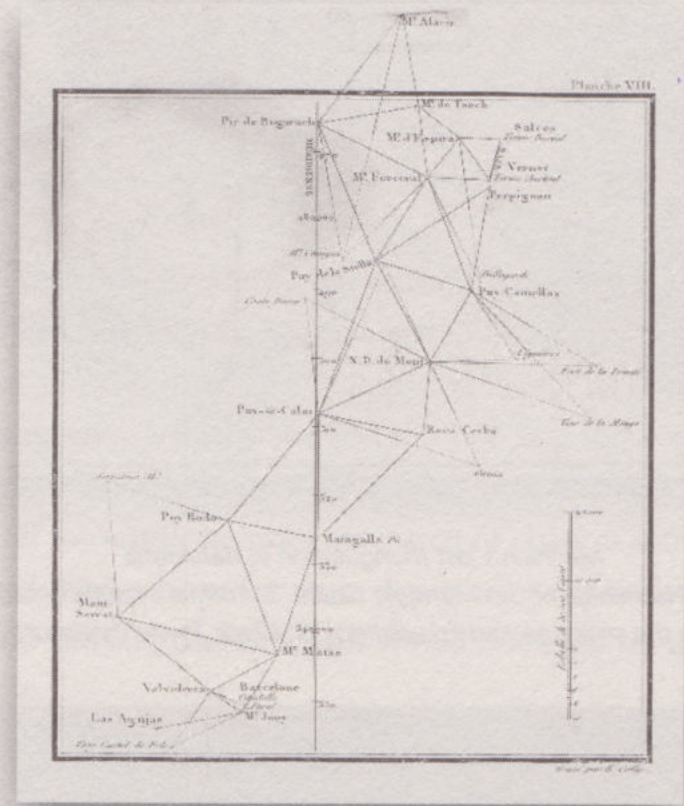
Méchain began his study of mathematics in Paris, but due to financial problems had to abandon them and offer his services as a tutor. From a very early age he devoted himself to research in the field of astronomy and geography and then dedicated his whole life to making detailed astronomical observations. This meticulous work was what caused him to be put in charge of measurement for the southern part of the meridian, from Rodez to Barcelona. Between 1771 and 1774 he worked with Charles Messier (1730–1817) on observing and recording celestial objects. Messier searched for comets and, with the aim of differentiating them from other celestial bodies, he began making a long list of objects, some of which were discovered by Méchain. The list began with some 40 objects and ended up having around a hundred, thanks also to the cooperation of other astronomers. This list is now known as the Messier Catalogue and is the delight of enthusiasts of astronomy as the objects described in it are visible with binoculars or small telescopes. Of course, it only includes celestial objects that are visible in the northern hemisphere, from the North Pole to a latitude of approximately 35°.

In 1793, the execution of Louis XVI led to a war between France and Spain, which also caused delays to the project.

Once the triangulation on Spanish territory was completed, Méchain intended to return to France but the military authorities would not allow him to do so, nor did they allow him access to Montjuïc Castle, from where he had previously carried out observations. He carried out more observations from the hostel in the street named Escudellers in Barcelona where he was lodging, and spotted a possible error in the measurement of 3.24" in the observations taken from Montjuïc; this possible error continued to be a worry to him till the day he died. His detention in Barcelona lasted from the June until the November of 1794, when he was finally allowed to return to France.

Like Méchain in Barcelona, Delambre also went through some difficult times in France. The inhabitants of the towns and villages through which the triangulation

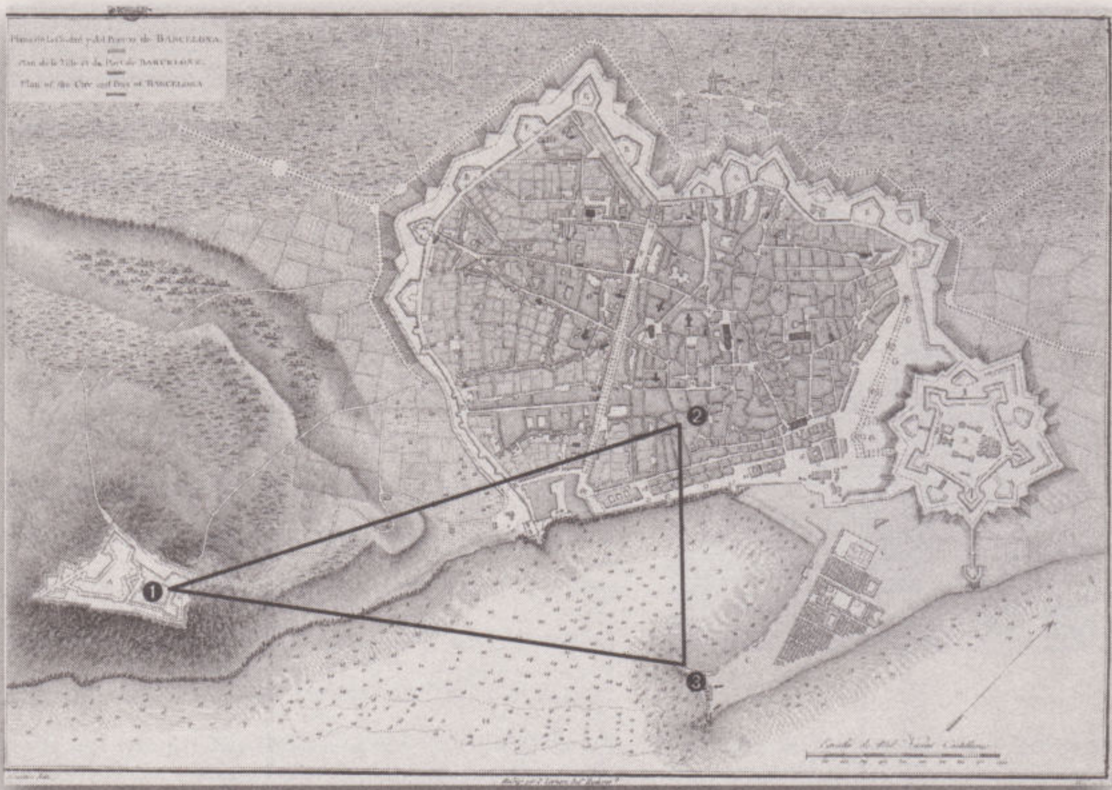
stretched tended not to understand what these gentlemen were getting up to when they signalled to one another in the middle of the night from bell towers and the tops of distant mountains. The scientists did this to avoid some errors in measurement that occurred during the day, but the local peasants took them for spies and often destroyed their signalling points or threw stones at the men.



The triangulation in Catalonia, from Méchain and Delambre's report.

The second expedition

There was a second expedition (1803–1804) to extend the measurement of the meridian to the Balearic Islands. On 31 August 1802, the Bureau des Longitudes decided to extend the measurements as far as Formentera in the Balearics. What were the reasons behind this decision? Firstly, it was believed that the longer the meridian arc, the more accurate the measurement would be. Secondly, the extension meant swivelling the meridian arc around 45° and minimising the error that could be produced by the flattening of the Earth. Méchain, who at that time was the director of the Paris Observatory, insisted that he should take part and direct the expedition, perhaps because of his obsession with the error in measurement he had spotted in Barcelona.



Méchain's last triangulation in Barcelona.
1. Torre del Homenaje at the Montjuïc Castle. 2. Former hostel Fontana de Oro (in the street named Escudellers). 3. Clock Tower (harbour).

The clock tower at the harbour in Barcelona, one of the vertices of Méchain's last triangulation.

Méchain was put in charge of the mission, and he set off on what would be his second journey to Spain. As his assistants he took the naval engineer Dezauche, his former pupil Jean-Baptiste Le Chevalier, who had spent a year in Madrid, and his 18-year-old son Augustin. Méchain arrived in Barcelona on 5 May 1803. There he received the additional help of Enrile, captain of the frigate named *La Prueba*, and of José Chaix, the deputy director of the Madrid Observatory. After long waits to attain the relevant permits to travel to the islands, he decided to look for new stations on the coast to the south of Barcelona and down to the area of El Montsià. During the autumn he carried out triangulation from Barcelona to different places, such as Garraf and Montserrat. He measured out five triangles on the coast.

At the beginning of November he was again in Barcelona and was finally able to obtain a passport, but when the frigate that was to take him to the islands arrived at the harbour in Barcelona to pick up Méchain and Enrile, its skipper and half of the crew were suffering from yellow fever and the ship had to be quarantined. Enrile decided to return to his ship, in spite of the fact that it was infected. Meanwhile, Le Chevalier set off for southern Spain (in search of classical antiquities) and Chaix headed for Madrid. The reasons for their departure were both fear of the yellow fever epidemic and the fact that Méchain would not even let them look through the repeating circle.

It was then that Méchain, to replace his assistants, signed on a Trinitarian monk named Agustí Canelles, who claimed to be an astronomer and was very confident of his skills and eager to play his part in an historic expedition. Canelles was appointed by the Spanish Government to cooperate in an official capacity with the French astronomer.

On 8 February 1804, Méchain was able to set sail for Ibiza. He stayed on both Ibiza and Majorca and set off afterwards for the mainland. In August 1804 he took measurements at El Puig, a mountain in the Desierto de Las Palmas, near Benicasim in Castellón de la Plana, where he had to stay on longer than expected as the monk Canelles had made some mistakes in the measurements.

This error in calculation resulted in a signal point being placed in the wrong position, and it cost them nearly two extra weeks of work. The indications are that it was at El Puig where Méchain caught the 'tertian fever' (malaria) which soon afterwards, on 20 September, was to cause his death in Castellon. Canelles also fell ill at the same time (with 'semi-tertian fever') and had to be bled three times.

A view of Benicasim from El Puig de Castellón.

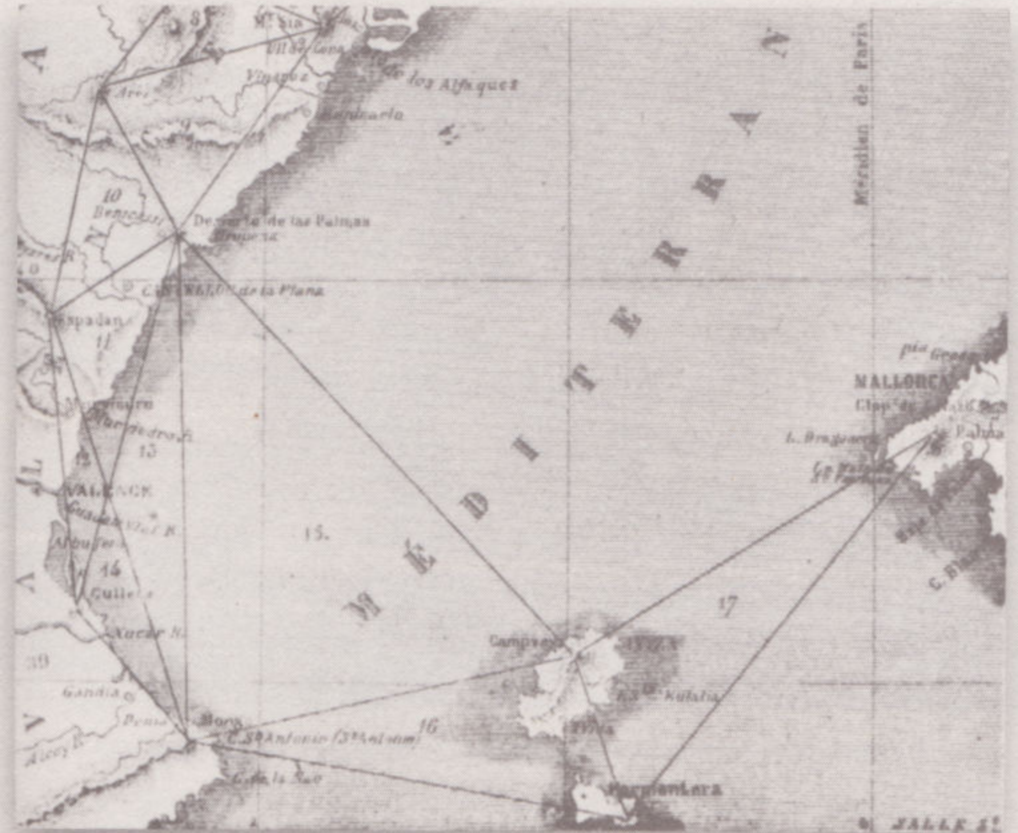
The third expedition

There was finally a third expedition (1806-1808) in which Biot and Arago took part. In 1805, the mathematician Laplace had put before the Bureau des Longitudes a proposal for continuing the measurements of the meridian as far as the Balearic Islands. The astronomer Jean Baptiste Biot and the young François Arago (born in Estagell, a village near Perpignan), who was the secretary of the Paris Observatory, were commissioned to carry on with the triangulation work initiated by Méchain and unfinished due to his premature death.

In September 1806, Biot and Arago started off on their journey to the Kingdom of Valencia. José Chaix (a Valencian astronomer and mathematician who was to become deputy director of the Madrid Observatory), who had already formed part of the previous mission led by Méchain, and José Rodríguez González (a Galician mathematician and professor at Santiago who was studying in Paris) were designated by the Spanish government to accompany them and take part in the geodetic operations. They also had the help of the naval lieutenant Manel Vacaro, who was in charge of the ship placed at the disposal of the scientists by the Spanish.

While they were carrying out observations in the desert of Palma, near the spot where Méchain had fallen ill, Jean Baptiste Biot also felt ill and asked for the help of Antoni Martí Franquès, a scientist from Altafulla (Tarragona), who took him in until he recovered. In January 1808, Biot returned to France with the result of 11 completed triangles and Arago now took over.

Arago's last project was to carry out an internal triangulation between Majorca and the Pitiusic Islands, which would mean that Majorca, Ibiza and Formentera had all been covered. The linking up of these three islands would enable a meridian arc of three degrees to be measured, which would contribute to completing the knowledge that they had on the shape of the Earth. In April 1808, Arago arrived in Majorca and set himself up at Mola de s'Esclop in order to carry out the final measurements.



The triangulation of the Balearic Islands.

On 27 May, just as Arago was making his final observations, news arrived in Majorca of the outbreak of the War of Independence, and this made things difficult for him. However, his adventurous spirit and the fact that he spoke Catalan saved him, as he recounts in his autobiography *History of My Youth*. News on the outbreak of the war had led some folk on the island to suspect that this Frenchman making

THE MONUMENT IN THE PLAZA DE LAS GLORIAS IN BARCELONA

A number of different cultural events were held to celebrate the 200th anniversary of the measurement of the Dunkirk–Barcelona meridian. For instance, in the square known as La Plaza de las Glòries in Barcelona they raised a monument donated by the city of Dunkirk, a city linked to Barcelona by the measurement of the meridian arc. The monument is a 50-metre-long steel wall that is 2 metres tall at its highest point. It is a scale reproduction of the imaginary line joining the two cities. At each end of the wall there a green marble plaque, one representing the Mediterranean Sea and the other representing the Atlantic Ocean. Three paragraphs inscribed on the sculpture give an explanation of the project in three languages: Catalan, Spanish and French.

The monument commemorating the measurement of the meridian arc between Dunkirk and Barcelona.

signals at night from the summit of s'Esclop was a spy, and they decided to capture him and hand him over to the authorities. Luckily for Arago, a sailor from his ship heard of the plot, warned him, and gave him sailor's clothing so that he was able to go unnoticed as he came down from the mountain, where he ran into an armed group of men who were on their way to arrest him. When he arrived back at the ship, José Rodríguez González, one of his colleagues appointed by the Spanish government,

asked the Admiral of the Fleet for help and the admiral had Arago interned in Bellver Castle until further notice. Some days later, Arago escaped from the castle, headed for Algiers, and from there managed to embark on a ship sailing for Marseilles. During the crossing, the ship was detained, and Arago was taken to the harbour at Palamós, from where he was transferred to Roses and spent some time there under arrest. Finally, on 30 August 1809 Arago was able to hand the Académie des Sciences the scientific report completing the calculations of all the measurements made.

The metre prevails?

The birth of the metre

The geodesic, astronomical and mathematical work carried out under the leadership of Delambre and Méchain led to the length of a metre becoming established as the universal standard measurement. While the two men's first measurement procedures were in progress, on 1 August 1793, the structure of the base-10 metric system was established and a provisional metre based on the previously made measurements was adopted.

On 22 June 1799, the National Assembly was presented with the platinum bar made by Lenoir, which measured the definitive metre (replacing the provisional metre) along with the standard which represented the kilogram – which was the mass of one cubic decimetre of distilled water at one atmosphere of pressure at a temperature of 3.98 °C. The standards were deposited in the archives of the French Republic.

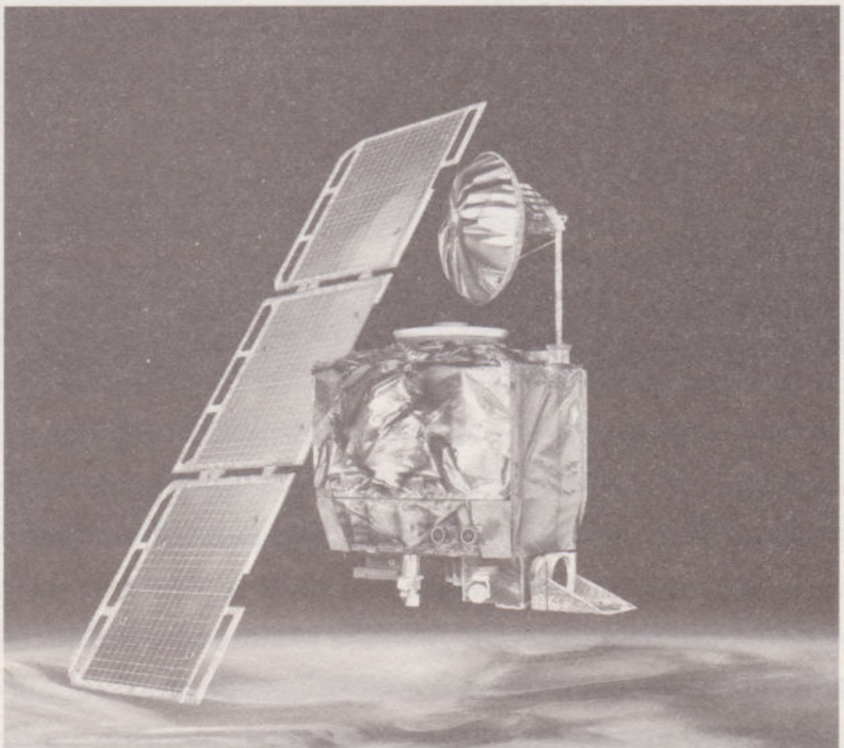
A comparison of the different results is contained in the minutes of the meeting of the Bureau des Longitudes on 30 August 1809, at which François Arago submitted the report on the measurements and calculations extending the meridian from Barcelona to Formentera. If the metre was to be defined in accordance with the new data, it would differ by less than 5 ten-thousands of a millimetre from the definitive metre approved in 1799 which had been recorded in the platinum bar. The small error did not result in a change in the platinum bar and it continued to be the standard for the metre until 90 years after its manufacture when it was replaced with another bar. The new bar was made in an X-shape of platinum and iridium (90% and 10%, respectively), which is a material that is more resistant to expansion and contraction. This bar is stored at the Bureau's headquarters at the Pavillon de Breteuil in Sèvres near Paris.

The metre had been born along with the metric system, but the adoption of these units by different countries was arduous and complicated. Belgium and Holland adopted the systems in 1816; Spain and Greece in 1849; Portugal in 1852; Germany in 1870; Austria in 1873; and Switzerland in 1875.

Two coexisting systems

One by one, almost every European country adopted the new metric system, but that was not the case in the United Kingdom, nor in the United States, which have both preserved their own systems.

On our travels by air we are told, on small screens, about the aircraft's flight path and altitude. The altitude is given in feet and, if we are lucky, in metres too. The number given in *feet* is always exact but, in contrast, the number in kilometres is not. For what strange reason does the captain make us fly at such a peculiar altitude? There is a very simple reason: all the plane's instruments measure in the Imperial system, feet for height and miles per hour for speed.



An image of the Mars Climate Orbiter *landing on Mars on 23 September 1999. The operation was never completed as two different systems of measure had been used, which led to the probe's destruction (source: NASA).*

A serious error in measurement

On 23 September 1999, NASA's *Mars Climate Orbiter* was due to land on Mars after a journey lasting 286 days. But NASA lost it 'simply' through a lack of standardisation of the units of measure they used. The *Mars Climate Orbiter* was wrecked due to a navigation error produced when the control team on Earth used Imperial units system to calculate the insertion parameters and sent data to the craft in those units, while the craft itself was using the metric system.

Let's look at the numbers of what happened. The minimum distance at which the craft could initiate its landing manoeuvre was 53 miles; otherwise it was likely to break up due to the high temperatures. The technicians initiated the manoeuvre at 59.54 miles, and so it seemed that everything was fine. For the craft, however, the 59.54 meant kilometres, and that was the root cause of the disaster: 59.54 is seemingly bigger than 53, but it turned out that it was 59.54 km, in other words, 37 miles – a long way below the limit of 53 miles for preventing the craft from disintegrating.

Chapter 6

Measuring in Modern Times

The old decimal metric system gave way to the International System of Units (SI) during the second half of the 20th century. The eagerness to measure the Earth, reveal its shape and be able to locate any point on it laid the foundations for modern geodesics and GPS measuring techniques. The need to draw up calendars and be able to use systems for measuring time led to accords being made on how time should be measured. Greek scientists' measurements of the skies and their first mathematical models of the cosmos evolved towards the modern theories on the Universe in which the huge interstellar distances made it necessary for new astronomical measures to be defined. Counting and measuring are two related activities that coexist in the physical world and in the mathematical model, but only in the latter can we speak of exact measurement related to the continuum and to real numbers ($\mathbb{R}$). By means of mathematics we can rectify, square and cube all actions that culminate in differential calculation, which laid the foundations for the theory of measurement to develop.

Diversity in methods of measuring

> "Some time ago I had a telephone call from a colleague concerning a student. My colleague had given the student zero points for his answer to a question on physics, but the student had complained that his answer was correct. Both of them had agreed to seek the judgement of an impartial person, and I was the person chosen. I read the question from the exam: 'Show how the height of a building can be calculated with the aid of barometer', and then read the student's answer: 'You take the barometer up to the top of the tower, attach a long rope is to it, and hang it down from the top of the building to the ground, and then make a mark on the rope. You then pull the rope up and measure the length to the mark. The result is the height of the building.'
>
> The student was right because he had answered the question correctly. But he could not be awarded top marks because in that case he would have got a degree in physics without having demonstrated that he has sufficient

knowledge in the field. I suggested he be given another chance. He would have to answer the same question again, but I warned him he had to demonstrate his knowledge of physics. Some minutes went by and the student had still not written anything. I asked him if he wanted to give up, and he answered that he had a lot of different answers to the problem and that he was trying to think of the best one.

After some time he answered: 'You place the barometer on the roof. You then make it drop down to the ground measure the time it takes with a chronometer. Afterwards, by applying the formula

$$x = \frac{gt^2}{2}$$

the height of the building can be obtained.' I asked my colleague if he agreed. He answered that he did, and he gave the student top marks. After leaving the office I came across the student again and asked him to tell me his other answers. 'Well – he replied – there are many ways to calculate the height of a building with a barometer; for example, you take the barometer out on a sunny day and you measure the height of the barometer, the length of its shadow and the length of the building's shadow. By applying a simple ratio this way we can also work out the height of the building.'

'OK – I said – what about the other ways?' He gave me different explanations, all of them valid and logical, but none of them was the one that my colleague and I had been hoping for. Finally, he concluded with: 'There are still other ways to solve the puzzle. The best one is probably to go to the caretaker's office, knock on the door and say to him: I've got a fantastic barometer here, and if you tell me the height of this building it's yours.' I asked the student if he knew the answer that I had been hoping for [the difference in pressure shown by a barometer in two places gives the difference in height between the two]. He admitted that he knew it, but that he was sick and tired of the university lecturers trying to teach him how to think."

Over time, this academic legend turned into a supposedly authentic anecdote attributed to the Danish physicist Niels Bohr (1885–1962) when he was a student

and the arbitrator was claimed to have been Ernest Rutherford (1871–1937). It is, in fact, not true, and the story stems from an article originally published in 1958 in the *Reader's Digest* magazine and written by Washington University's physics professor Alexander Calandra (1911–2006), who was very interested in the didactics of science.

The story shows how the measurement of a certain physical quantity can be calculated by very diverse methods. But although there is a diversity of methods, both direct and indirect, for arriving at an accurate measurement, it ought ideally be expressed in universally accepted units, as was the wish of 18th-century scientists.

Measuring in a physical world

The International System of Units

In 1875 an international agreement called the Metre Convention was reached with the aim of preserving the metric system standards. One of the organisations created by this convention was the General Conference of Weights and Measures, which is its decision-making body. Its first meeting took place in 1889. Nowadays it meets every four years. More than 50 countries adhere to the Treaty of the Metre.

In 1960, in accordance with an agreement signed by 36 nations, the General Conference of Weights and Measures adopted the International System of Units (SI), which modernised the metric system. A system of units is a set of units of measure in which each quantity has only one single unit associated to it. The standard of each unit must comply with three conditions:

—Stability (it must not vary over time or be caused to vary by the person doing the measuring).
—Universality (it can be used in any part of the world).
—Reproducibility (it can be easily reproduced).

Seven basic quantities are defined in the SI. All other physical quantities derive from these seven and can be expressed by means of a mathematical combination of them. Each quantity has an associated unit. The base units correspond to the base quantities, and the derivatives of the units reproduce the same calculations contained in their definitions. The table below contains the SI's base quantities and the units in which they are expressed:

Base physical quantity	Base unit	Year of unit definition	Symbol
Length	Metre	1983	m
Mass	Kilogram	1889 (1901)	kg
Time	Second	1967–1968	s
Electric current intensity	Ampere	1948	A
Temperature	Kelvin	1967–1968	K
Amount of substance	Mole	1971	mol
Luminous intensity	Candela	1979	cd

DEFINITIONS OF THE SI BASIC UNITS

—The **metre** (m) is the distance travelled by light in a vacuum in 1/299,792,458 seconds.

—A **second** (s) is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between two energy levels of the ground state of the caesium 133 atom.

— The **kilogram** (kg) was defined in terms of the Planck constant (h), the speed of light (c) and hyperfine transition frequency of ^{133}Cs ($\Delta\nu_{Cs}$), as approved by the General Conference on Weights and Measures (CGPM) on November 16, 2018, as $(917097121160018 / 621541050725904 75)10^{42}\ h\ \Delta\nu_{Cs}/c^2$. *(Wikipedia)*

—The **ampere** (A) is the constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in a vacuum, would produce a force equal to $2 \cdot 10^{-7}$ newtons per metre of length.

—The **kelvin** (K) is the fraction 1 / 273.16 of the thermodynamic temperature of the triple point of water.

—The **mole** (mol) is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12 (approx. $6.02214179 \cdot 10^{23}$ elementary entities – atoms, molecules, ions, electrons, radicals or other particles).

—The **candela** (cd) is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency $540 \cdot 10^{12}$ hertz and that has a radiant intensity in that direction of 1/683 watt per steradian.

Geodesics, chronometry and astrometry

Aided by modern technology, present-day geodesics, chronometry and astrometry use sophisticated methods of measuring. In some cases their results have on impact on everyday activities, for instance, on cars' GPS systems and mobile telephony. GPS,

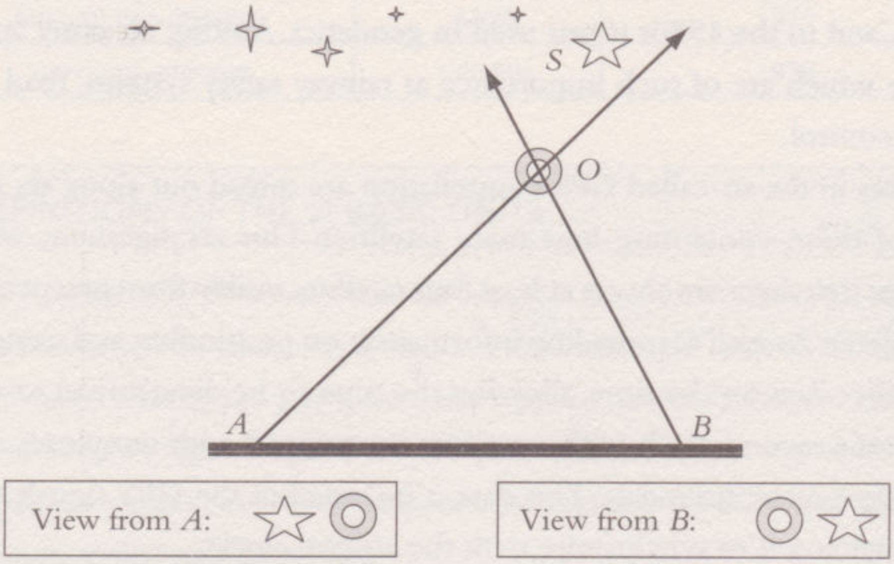
which stands for *Global Positioning System* and was developed in the USA, presents the longitude, latitude and altitude of an object (for instance, a person, a car or a ship) with an accuracy of mere centimetres. This is achieved thanks to a network of 24 satellites (plus three more back-ups) that orbit the Earth at an altitude of 20,200 km, covering the whole of the terrestrial surface by means of a programming system that synchronises the satellites' paths. Though a few decades ago it was largely restricted to military applications, maritime and aerial navigation, GPS has spread to civilian use, and in the 1980s it was used in geodetics. Among its many applications are some which are of such importance as railway safety systems, road traffic and air traffic control.

The satellites in the so-called GPS constellation are spread out along six orbital planes, each of them containing four main satellites. This arrangement with 24 satellites ensures that there are always at least four satellites visible from practically any point on the planet. As well as providing information on positioning and navigation, GPS also supplies data on the time, allowing the time to be determined to within 100 billionths of a second. Each of the satellites is equipped with numerous atomic clocks with very precise time data. This data is included in the GPS signals sent to the receiver, enabling it to synchronise with the atomic clocks.

The basic idea for determining a position is to calculate distances from the satellites to the GPS receivers by measuring time. The receiver automatically localises a minimum of three satellites, from which it receives signals showing each of the satellites' positions and clock. The GPS receiver sychronises its clock and calculates the delay in the signals, and thus discovers the distance to each satellite. The triangulation (inverse trilateration) is not so much based on calculating angles with respect to known points but on determining the distance from each satellite to the receiver, which allows the receiver to determine its position relative to them. By knowing the coordinates of each one from the signal they emit, the real coordinates (latitude and longitude) of the receiver (its absolute position) can be determined. To calculate an altitude it is necessary to use the signal from a fourth satellite.

Artificial satellites are also indispensable in astrometry for measuring and studying the position and distance of celestial bodies with accuracy. Satellites such as *Hipparcos* (an acronym for *High-Precision Parallax Collecting Satellite*), which the European Space Agency launched in 1989, have enabled parallax and the motions of millions of stars to be measured. Parallax can be defined as the angle formed by the two directions of two lines of sight aimed at the same point (at a star, for instance) from two different points A and B which are sufficiently distant from each other and not aligned with

that point. It was historically very difficult to detect on account of the huge distances between us and the stars, even the closest ones. Such great distances mean that the lines of sight are practically parallel. It was not possible to determine the parallax of a star until 1838 when the astronomer and mathematician Friederich Wilhelm Bessel (1784-1846) managed it with the star known as Cygni 61.



An object O *will be observed to the left or to the right of a star* S *taken as a reference on the starry background.*

The nearest star to the Earth is Proxima Centauri, which has a parallax of 0.765"; this shows that stellar parallaxes are always under one arc-second. The greater the distance, the less the parallax, and therefore errors become progressively more significant. In these cases, measurement is taken of the electromagnetic radiation spectrum emitted by the astronomical object being studies. This allows the displacement taken place over the distance to be calculated, from which it is possible to calculate how far away the object is from us. Working with such enormous distances has led to the use of special units that are much larger than the metre for astronomical measurements.

The *astronomical unit* (AU) is a unit of length that is approximately equivalent to the distance between the Earth and the Sun. It is defined as 149,597,870 km and is used for measurements within the Solar System. The definition of the *parsec* or *pc* is the distance that produces a parallax of one second (1") of arc. A *light-year* (the

SOME ASTRONOMICAL DISTANCES

The *astronomical unit*, the *light-year* and the *parsec* are the units which enable distances within the Solar System, the Milky Way and intergalactic space to be efficiently expressed. Rounding up or down, we get the following equivalences: 1 parsec = 3.26 light-years = 206.265 AU = = 30,875 trillion kilometres. The mean distance from the Earth to the Sun is around 150 million kilometres, approximately one astronomical unit (1 AU). The light we see from the Sun takes 8.32 minutes to reach the Earth; we therefore say that the Earth is at a distance of 8.32 light-minutes from the Sun.

Some distances to the Sun:

Heavenly body	Approximate distance
Venus	Less than 0.68 AU
Earth	1 AU = 8.32 light-minutes
Jupiter	More than 5.2 AU
Pluto	39.5 AU
Centre of the Milky Way	8,500 pc ≅ 30,000 light-years ≅ ≅ 1,753 million AU

Some distances to the Earth:

Heavenly body	Distance	Particular characteristic
Moon	0.0026 AU	The Earth's only natural satellite
Proxima Centauri	4.2 light-years = 270,000 AU	The nearest star
Sirius (star)	8.6 light-years = 540,000 AU	Its first appearance on the horizon before dawn and after a long period of being invisible marked the beginning of the ancient Egyptian calendar.
Andromeda Galaxy (M31)	2.56 million light-years = = 775 kpc	The object most distant from the Earth visible with the naked eye

distance travelled by light in one year) is defined as the distance that a photon would travel in a Julian year (365.25 days of 86,400 seconds) at the speed of light in a vacuum (299,792.458 km/s) at an infinite distance from any gravitational or magnetic field.

Measurement in the physical world uses rational numbers and leads to an approximate value of the result. In the mathematical model, where measures are made with real numbers, the formalisation of the concept of measuring gave rise to the theory of measurement. As we've said, rectifying, squaring and cubing played an important role in this field.

Measuring in the mathematical model

Rectifying

The word rectify (from the Latin *rectificāre*; from *rectus*, 'straight', and *facĕre*, 'make') has many entries in the dictionary, including a geometrical one: "Find a straight line the length of which is equal to that of a given curve." A graphic interpretation leads to an approximate calculation method, which could be: the curve is divided into the smallest possible chords, one following on from the other, and finally the lengths of them all are added together.

Division of the curve into chords.

This method is reminiscent of the one a teacher used with his secondary school pupils to measure the length of a coastline. He would explain: "To carry out the measurement we need to get hold of a scale map, a sufficiently long thread, and a graduated ruler. The thread needs to be dampened a little so as to make it more flexible, and then it is placed along the coastline by following all its contours. Once all the coastline has been covered with the thread, the only thing that remains to be done is to place it beside the ruler and measure its length. Finally, the scale of the map provides the data for converting the length and finding the real measurement."

Since ancient times, one of the best-known rectifications has been that of a circle. Ancient Egyptian mathematicians established a correct rule for finding the circumference of a circle. The rule stated that the ratio of the area of a circle to its circumference is the same as the ratio of the area of the square circumscribed to its perimeter. This observation is a precise geometric relationship ($r/2$, with r being the radius of the circle), while the numerical value of π used by them was only an approximation (3.16, or 3 and one sixth).

Rectifying, squaring and cubing are also present in the *Jiuzhang suanshu,* also known as *The Nine Chapters of the Mathematical Art,* a classic text on ancient Chinese mathematics compiled in the 1st century AD. In the first of the nine chapters, entitled *Rectangular Fields*, the number π is calculated through a process of *exhaustion*: a hexagon is inscribed on the circumference and its length is compared with the perimeter of the polygon; and thus the result is $\pi = 3$. By continuing the process with polygons of 12, 24, 48 and 96 sides the end result is $\pi = 3.1410240$.

In the mathematics of ancient India, rectification is also used for the calculation of the length of a circumference. In Chapter II of his work entitled *Aryabhatiya*

A RABBIT CROSSING THE MERIDIAN

Here is a curious puzzle about a circumference and radius that produces an unexpected answer and has a bearing on our story of the history of the meridian. Remember that a meridian measures approximately 40,000 km. If we imagine the Earth to be completely spherical, a rope stretching right around the Earth would measure those 40,000 km. If we were to make that rope just one metre longer and have it encircle the Earth, would a rabbit be able to pass underneath it? Although 1 metre may seem very little compared to 40,000 km, the answer is yes. But the most surprising part is that the increase in the radius of the new circumference is always the same if the circumference is extended by 1 metre, whatever the initial radius may be. We can confirm the answer just by making some small calculations:

Let r_1 be the initial radius and the length of the circumference $L_1 = 2\pi r_1$, when the length of the circumference is extended 1 metre, the new length of the circumference is $L_2 = 2\pi r_1 + 1$; this new circumference has radius $r_2 = \frac{2\pi r_1 + 1}{2\pi}$, that is to say that $r_2 = r_1 + \frac{1}{2\pi}$, which enables us to see that whatever the initial radius, by extending the length of the circumference by 1 metre, the repercussion on the new radius is $\frac{1}{2\pi}$m. Going back to our initial puzzle and by making the conversion to centimetres, $\frac{100}{2\pi}\text{cm} \approx 15.91549431\,\text{cm}$, a rabbit will surely be able to pass through that gap – and with ease!

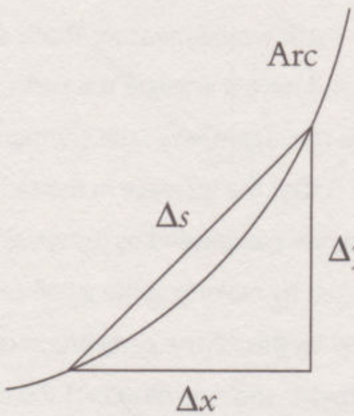
(c. 500 AD), Aryabhata arrived at an approximation of $\pi = 3.1416$. He calculated the perimeter of an inscribed regular polygon of 384 sides by using a method of *exhaustion* similar to the one in the *Nine Chapters* in Chinese mathematics, but with two further steps: he also began with the hexagon and doubled every side up to 384 (6, 12, 24, 48, 96, 192, 384).

Since ancient times, in different cultures attempts have been made (employing a variety of methods) to match curves to straight segments of the same length. Even in the 17th century, contests were held to determine lengths of arcs on specific curves, such as the Archimedes spiral, the catenary or the cycloid. The calculations were based on geometric methods. At the end of the 17th century, the definitive step forward was taken by differential calculus. In differential calculus the length of an arc was defined by the formula

$$S = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\, dx,$$

where $f(x)$ is the function of which the length is required, $f'(x)$ is its derivative and both functions must be continuous in the interval $[a,b]$. In these conditions, S is the length of the arc bounded by a and b.

The proof of this formula of founded on the initial idea of covering the curve with a chain of small rectilinear segments and applying Pythagoras' theorem to each of them. Each segment is of length $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$.



An arc with segment Δs, the hypotenuse of the triangle with sides Δx and Δy.

An approximation to S would be given by the sum of all the hypotenuses:

$$S \approx \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}, \text{ equivalent to: } S \approx \sum_{i=1}^{n} \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i .$$

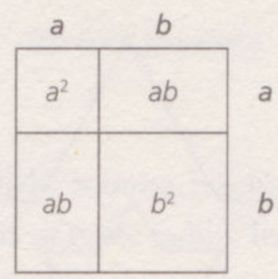
The smaller the n segments are, the better the approximation will be. On the limit, each Δx_i tends to zero, and by applying the definition of integral between a and b provided by differential calculus, the result is the formulated ratio:

$$S = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\, dx.$$

Squaring

The verb 'to square' has two definitions in a mathematical sense. The first one is geometric: "Determine or find a square equivalent in area to a given figure"; the second one, relating to calculation, is "Raise a number or algebraic expression to the second power, that is, multiply it by itself." The two mathematical definitions of the word square are very much related to each other: "Raise a number or algebraic expression to the second power" can simply be interpreted as indicating the calculations required to obtain the measure of the area of a square surface the sides of which is the number or the algebraic expression that is to be raised to the second power.

The figure below shows how, when raising $(a+b)$ to the second power, it gives the resulting area of the square of side $a+b$:



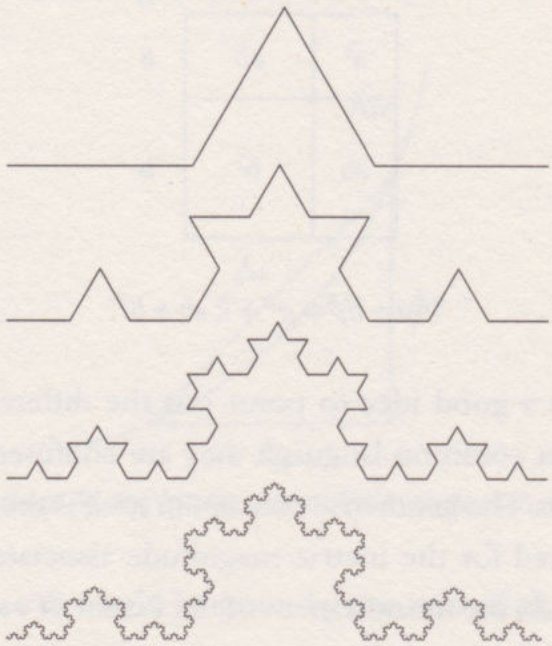
$$(a+b)^2 = a^2 + 2ab + b^2.$$

At this point it is a good idea to point out the difference between area and surface as, although in common language they are confused, they are two distinct mathematical elements. The geometric concept in itself is the surface, while the term area should be reserved for the metric magnitude associated with the geometric concept, in other words, the measurement of the extent of a surface expressed in the corresponding units.

Two other terms that are sometimes confused are perimeter and area. Perimeter can be differentiated from area in the same way that we differentiated between

circumference and circle. Perimeter, from the Latin *perimĕtros*, which in turn is from the Greek περίμετρος, is the outline of a surface or figure, while the area is the measurement of the surface or, put another way, it is the measurement of the interior of the perimeter of the surface. In this context we can ask ourselves the following: starting from a fixed perimeter – for example, from a string of a given length – what is the rectangle of largest area that can be bounded by that string? And in a wider sense, what is the figure of largest area that can be bounded by that string? The answer to the first question is a square, and to the second, a circle. These results have been known since ancient times and have numerous applications in real life. In Chapter 1, mention was made of the fact that traditional homes from a wide range cultures (Inuit, American Indian and Kenyan, for instance) were circular so as to achieve the maximum surface with the least material.

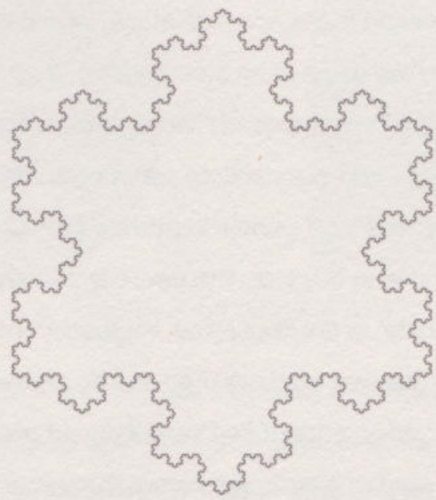
A paradoxical case concerning the relationship between area and perimeter is the one shown by certain figures which, despite having a finite area, have a surrounding perimeter that is infinite. This is the field of fractals, which includes, for instance, the *Koch snowflake* is a curve that is continuous but non-differentiable at any point. It was described by the Swedish mathematician Helge von Koch (1870–1924) in 1904. From four segments of the same length (for instance, of value 1), connected as shown in the first figure of the series below (K_0) and by means of a repetitive process, the Koch curve can be built, which is the step prior to constructing the snowflake.



The first four sequences in the construction of the Koch curve.
From the top: K_0, K_1, K_2 and K_3.

In the first iteration we replace each of the 4 segments of K_0 with a copy of K_0 reduced by a factor of 3, and we use K_1 to denote the result, which is composed of $16 = 4^2$ segments. Then we replace each of the 16 segments with a copy of K_0 reduced by a factor of $9 = 3^2$, and we use K_2 to denote the result, which is composed of $64 = 4^3$ segments, and so on. The Koch curve is the result of taking the limit of the sequence K_i when i tends to infinity.

To construct the Koch snowflake we take 3 copies of K_0, form an equilateral triangle and replace each of its three sides by the curves described. The resulting object, the Koch snowflake, can be seen in the following diagram:



The Koch snowflake.

The area of the Koch snowflake is finite, but its perimeter is infinite. The area is finite because it fits inside a circle of finite radius. In the example, where the original segments of K_0 were of length 1, it can be proven that the snowflake fits in a circle of radius 3. To prove that the snowflake's perimeter is infinite, it is sufficient to see that the Koch curve is infinite. To do that, we shall calculate the length $l(K_i)$ at each of the steps of its construction. The length of K_0 is 4 (4 sides of length 1). As K_1 is composed of $16 = 4^2$ segments of length 1/3, its lenth is:

$$l(K_1) = \frac{4^2}{3}.$$

By generalising the process we get:

$$l(K_n) = \frac{4^{n+1}}{3^n} = 4\left(\frac{4}{3}\right)^n \text{, and so }, l(K) = lím_{n \to \infty} l(K_n) = \infty.$$

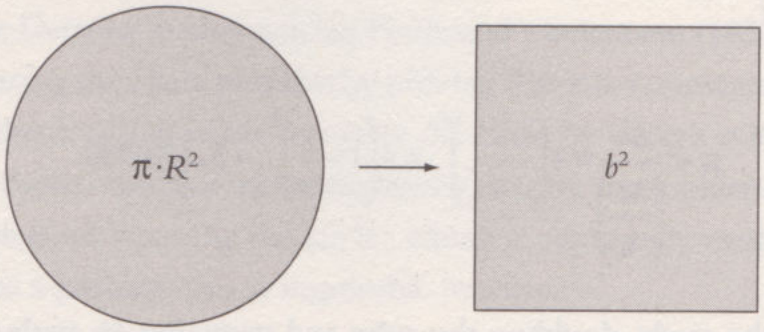
THE MAXIMUM AREA BOUNDED BY AN OXHIDE

Mutto, the King of Tyre, had two children, Pygmalion and Alyssa (the Tyrian name of Queen Dido). On his death, the throne passed to his son, the young Pygmalion, who was still a child. Urged on by Pygmalion, Alyssa married her uncle Sychaeus – a priest of Hercules and second only in power to the king – because Pygmalion wanted to get his hands on treasure that Sychaeus had hidden away. Some time later, Pygmalion tried to persuade Alyssa to find out where her husband's treasure was hidden, which she did, though she did not tell him the true hiding place. Pygmalion had Sychaeus murdered so as to get hold of his wealth, but his sister managed to flee by ship, taking the treasure with her and accompanied by a number of Tyrian nobles. The fugitives disembarked in the north of Africa, where they were well received by the local inhabitants with whom they came to an arrangement: they could settle there but occupying only the land that an ox-hide encompasses. The newcomers cut an oxhide up into very fine strips and joined them together and were able to surround quite a considerable piece of land, where they founded the city. The local people kept their part of the agreement and let them have the land that was bounded in this way. The new city was given the name of Byrsa, which means 'mountain of the ox-hide' in the Phoenician language. Some time later, Iarbas, the king of a neighbouring township, wanted to marry Dido and threatened to declare war on the fugitives if she refused. Dido refused and then killed herself. Based on this legend, Virgil constructed the Aeneid in which we can read of how Trojan hero Aeneas was forced onto the African coast by a storm and taken in by the inhabitants of Carthage, the city founded by Dido. In Virgil's story, Dido falls in love with Aeneas and begs him to stay; when he rejects her, she kills herself.

Aeneas telling Dido about the sacking of the city of Troy.
An oil painting by the French artist Pierre-Narcisse Guérin.

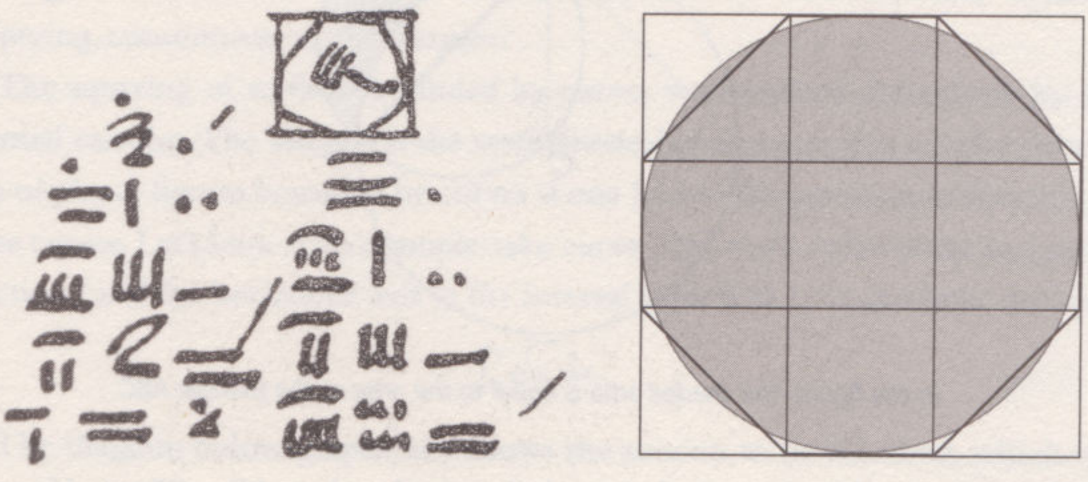
In our preamble about rectification, we mentioned measuring a coastline by using a dampened thread on a map. A similar process can be followed to calculate the area of a surface. In this case a sheet of transparent graph paper is needed. Counting the number of squares covering the surface and using the map's scale will give a good approximation of the area being measured.

Since antiquity one of the best-known quadratures has been squaring the circle. It consists of finding – just with a ruler and pair of compasses – a square that has the same area as that of a given circle.



The area of a circle must become the area of a square.

The Ahmes Papyrus (also known as the Rhind Papyrus because it was acquired by Henry Rhind in 1858 after being found in a building in Luxor) was written by the scribe named Ahmes around 1650 BC but it contains writings and knowledge from the period 2000 BC–1800 BC. In problem 48, the surface of a circle with a diameter of 9 units is identified with that of a polygon of 8 sides inscribed in a square of side 9 units, as shown in the diagram:



A fragment from the Rhind Papyrus
and the 8-sided polygon analysed.

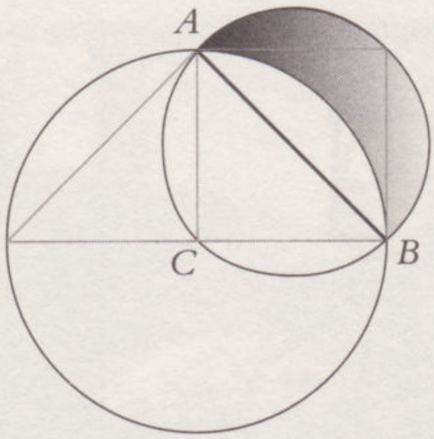
The area of the circle is: $\left(\frac{9}{2}\right)^2 \cdot \pi$.

The area of the polygon approximates to 64. It is really 63, as each square has area $3 \times 3 = 9$, and the polygon is formed by 5 whole squares and four half squares, which makes a total of 7 squares with area 9. For the calculations, area 64 is taken as an approximation since it is a perfect square, 8^2.

Thus, $\left(\frac{9}{2}\right)^2 \cdot \pi \approx 64$, and by carrying out calculations and simplifications the result is:

$$\pi \approx \frac{64}{\left(\frac{81}{4}\right)} = 4\left(\frac{9-1}{9}\right)^2 = 4\left(1-\frac{1}{9}\right)^2 = 3.1604938.$$

Squaring the circle, doubling the cube and trisecting an angle were the three classical problems in ancient Greek mathematics. The possibility of squaring planar surfaces bounded by curves, and in particular, squaring the circle, would not have seemed so plausible to the Greeks if it had not been for the fact that Hippocrates of Kos (c.470–c.410 BC) demonstrated that certain curvilinear figures called lunes that he had purposely constructed could be squared.



In the figure, the shaded area is equal to the area of the triangle ABC.

So as to simplify the calculations, let's consider that $AC = CB = 1$. If we can manage to see that lune AB which completes triangle ABC and converts it into a quadrant of the larger circle is equal to the two lunes that complete the triangle and

convert it into a half circle that has diameter AB, the statement will be proven. It will be enough to observe that, in the smaller circle, the triangle plus the two lunes are a half circle, and the shaded area plus the lune also is.

The larger circle is of radius 1; therefore its area will be π. A quadrant has area $\pi/4$. The smaller circle has diameter $\sqrt{2}$, radius $\sqrt{2}/2$ and an area of $1/2\pi$. Half the small circle has again an area of $\pi/4$, in other words, half the small circle is like a quadrant of the large circle. In these conditions we can state that the two small lunes have the same area as the larger one, from which it can be deduced that the triangle is equal to the shaded area.

In 1882 the German mathematician Ferdinand Lindemann (1852–1939) put an end to the squaring the circle problem by proving that π is a transcendental number, which means that it is impossible to square the circle by using a ruler and a pair of compasses. All those centuries spent attempting to solve the problem probably gave rise to the expression 'squaring the circle', which is commonly used in a figurative sense to refer to a problem that is impossible to solve.

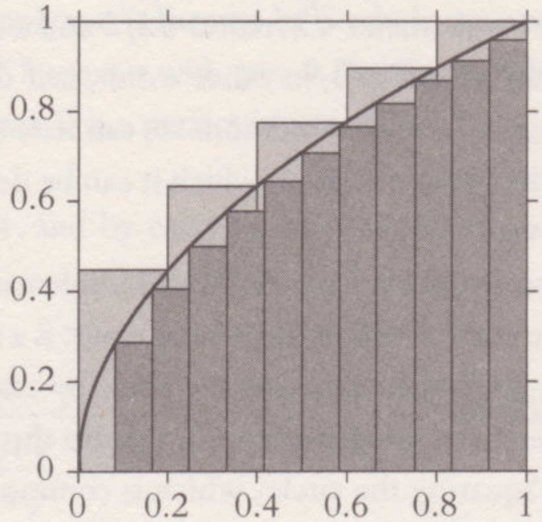
Eudoxus also worked on the issue of squarings. His method was a geometric-mathematical procedure of approximating a result the accuracy of which is increased by progressing a series of calculations. In real terms, this method is similar to the one used in China and India to calculate the perimeter of a circumference or the area of a circle by means of successive approximations using polygons. Later, Archimedes would also use it for calculating the area under the arc of a parabola and the volume of a sphere. All these proofs were included by Euclid in Book XII of his *Elements* (c.300 BC). In the 17th century, Gregory of St. Vincent (1584–1667) gave this method the term 'exhaustion', from the Latin *exhausti*, meaning 'emptying, consummation, termination'.

The squaring of surfaces bounded by curves was resolved definitively by differential calculus. The integral is the mathematical instrument that enables calculation of planar figures bounded by curves if one knows the equation or formula for those curves. Let's look at an example: take curve $f(x)=\sqrt{x}$; what is the area under function f and the horizontal axis in the interval from 0 to 1? In symbolic notation:

$$\int_0^1 \sqrt{x}\,dx.$$

The diagram below graphically shows the process to be followed, which will end by tending to the limit. It is a case of constructing a series of rectangles which, as in the case of the graph paper and the map, enable an approximate value for the area to be calculated by adding the areas of the rectangles. These

rectangles can be built above (rounding up) or below (rounding down) the curve being squared.



A diagram with five partitions for the rounding up and twelve for the rounding down (in this case the first rectangle is not visible because it has zero height).

The fundamental theorem of calculus, as developed by Newton and Leibniz, is the fundamental link between the operations of derivation and integration. By applying it to the curve bounding the area being calculated $f(x)=\sqrt{x}$, we can obtain the primitive function

$$F\left(x\right)=\frac{2}{3}x^{\frac{3}{2}},$$

which must be evaluated between the values of the extremes, 0 and 1 in this case, and calculate the difference, $F(1)-F(0)$. Thus, the exact value of the area under the curve is calculated formally as:

$$\int_0^1\sqrt{x}\,dx=\int_0^1 x^{\frac{1}{2}}\,dx=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1=\frac{2}{3}.$$

Cubing

According to the dictionary, to cube is "to measure a volume" and also "to raise a number or algebraic expression to the third power, that is, multiply it by itself twice". The two definitions are related to each other if the second one is interpreted as the calculations required to find the volume of a cube the edge of which is the number

or algebraic expression mentioned in the definition. The dictionary also provides a reference to the term *cubic*, from the Latin *cubĭcus,* this being from the Greek κυβικός.

If squaring the circle was the classical problem of ancient Greek geometry relative to squaring, doubling the cube takes on the same role in the case of cubing. A traditional legend tells of how an outbreak of bubonic plague that occurred in Athens around 428 BC so terrified the populace that the city leaders had to call on the help of the god Apollo to put an end to the epidemic. Apollo's oracle at Delphi told them that they would have to build an altar with a volume that was double that of the altar of Apollo in the temple. Although the epidemic of the plague eventually came to an end, the attempts to construct an altar with double volume ended in failure.

In one of his works, Euripides dramatised the problem of doubling the cube through the tale of King Minos, who when building the tomb of his son Glaucus, declared that a cubic mausoleum which only measured one hundred feet along its sides was a very small space for a king's tomb, and he ordered that it be doubled by doubling each side and retaining its cubic shape. This was a serious mistake by Minos: if the side of a cube is doubled the result is a cube with a volume eight times that of the original.

Without the benefit of modern algebraic notation, the ancient Greek mathematicians had to solve the problem just with a straightedge and a pair of compasses. To find a side x of the square of area double that of side a they inserted a median proportion between a and $2a$:

$$\frac{a}{x}=\frac{x}{2a}, \text{ then } x=a\sqrt{2}.$$

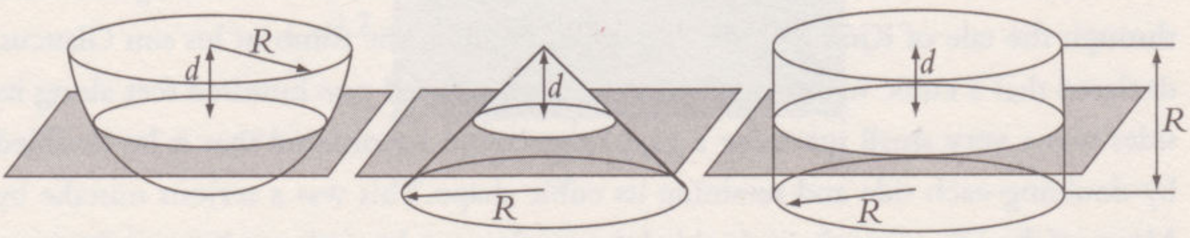
The problem of doubling the cube was approached with the same idea and it was seen that solving the problem was the equivalent of determining two median proportions between a and $2a$. Doubling a cube of side a consists of determining side x of a cube whose volume is $2a^3$. [$x=\sqrt[3]{2a^3}=a\sqrt[3]{2}$.] If two values x and y can be found that are median proportions between a and $2a$, as indicated by the following equality

$$\frac{a}{x}=\frac{x}{y}=\frac{y}{2a},$$

then the problem will have been solved. This way of working is very typical of mathematics, i.e. the requirement to translate the original question into another that is seemingly easier to solve. The problem now is how to construct these two median proportions.

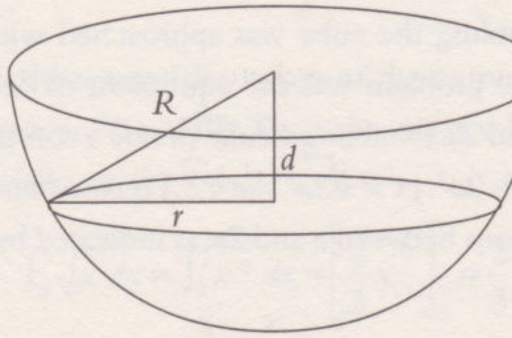
As well as doubling the cube, there were other calculations of volumes that preoccupied Greek mathematicians. According to Archimedes, Eudoxus proved that the volume of a cone is a third of the volume of a cylinder which has the same base and height. Archimedes proved that the area of a circle is the same as that of a right-angled triangle with one leg with a length of the radius and the other leg the length of the circumference. He also calculated the volume of a sphere from the volumes of the cylinder and the cone.

How did he get the volume of a sphere? He started from a hemisphere of radius R and at its side placed a right cone and a right circular cylinder, both also having bases of radius R:



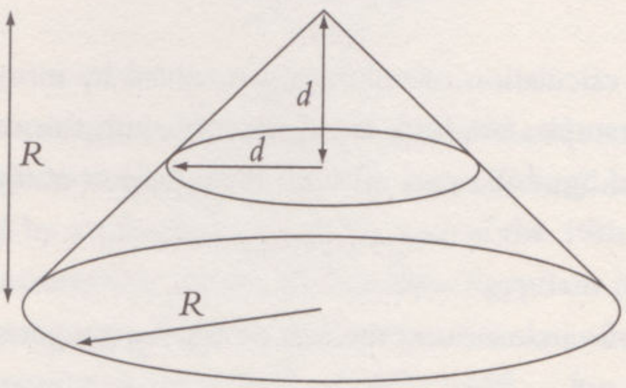
A diagram showing the cuts in the hemisphere, the cone and the cylinder.

He cut the three figures with a plane parallel to the base of the cylinder (at the same distance d from the upper part of the three figures) and studied the sections that the plane would create in each of them. On the cylinder, a circumference of radius R; on the hemisphere, also a circumference, but with a different radius, let's say r.



The figure shows the relationship between r, d *and* R *in the hemisphere.*

In accordance with Pythagoras' theorem, it holds that $r^2 + d^2 = R^2$. In the cone, the section is also a circumference, but now of radius d, because the radius of the opening of the cone is 45°.



The figure shows the relationship between R *and* d *in the cone.*

The areas of the three sections are calculated and the result is:

Figure	Area of a section
Cylinder	πR^2
Hemisphere	πr^2
Cone	πd^2

And from the fact that $r^2 + d^2 = R^2$, the result is that:

Section of the cylinder = Section of the hemisphere + Section of the cone.

The sections in the figure are like slices of bread; if we have the ratio for each slice, it seems quite clear that the volumes follow the same ratio. Hence:

Cylinder volume = Hemisphere volume + Cone volume.

But Archimedes knew the volumes of the cylinder and the cone:

$$V(\text{cylinder}) = \pi R^3 \qquad V(\text{cone}) = \frac{1}{3}\pi R^3.$$

Thus he was able to establish the equality

$$V(\text{hemisphere}) = V(\text{cylinder}) - V(\text{cone}) = \pi R^3 - \frac{1}{3}\pi R^3 = \frac{2}{3}\pi R^3,$$

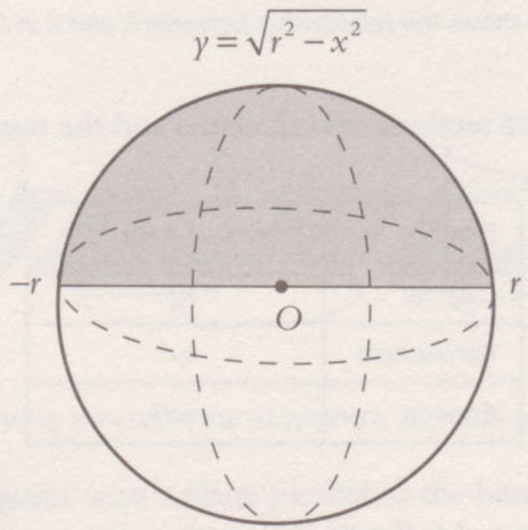
and, finally,

$$V(\text{sphere}) = \frac{4}{3}\pi R^3.$$

Once again, the calculation of volumes was solved by introducing differential calculation. As an example, let's look at calculations with this method for the volume of a sphere of radius r. We start off with the equation of the circumference

$$x^2 + y^2 = r^2.$$

By rotating a semi-circle around the axis of abscissa we get a sphere.



A sphere generated by rotating a semi-circle.

If a figure between $y = f(x)$, $y = 0$, $x = a$ and $x = b$ is rotated around axis OX, the volume of the solid generated is given by the formula:

$$V = \pi \int_a^b f(x)^2 dx.$$

This formula is reminiscent of Archimedes' calculations if we take $\pi f(x)^2$ as the area of the circle obtained and imagine the solid generated by revolution, as in the case of Archimedes, of sections like 'slices' and remember that $\int_a^b \ldots dx$ is the symbol used to denote the integral, that is to say, the sum of the infinite sections of infinitesimal thickness (dx) which make up the volume of the solid of revolution. Thus, in our example:

$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_{-r}^{r} \left(r^2 - x^2\right) dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \pi \left(\frac{2r^3}{3} + \frac{2r^3}{3} \right) = \frac{4}{3}\pi r^3.$$

Epilogue

The concept of measuring is more than 5,000 years old and came from the need to gauge the size of objects in people's immediate surroundings. Let's look at some of the challenges faced by mathematicians at the end of the 19th century which led them to develop measurement theory. The ancient Egyptians (the Rhind Papyrus, the Moscow Papyrus) worked with calculations of areas and volumes and came to work with approximations of $\pi \cong 4\,(1-1/9)^2 = 3{,}160\ldots$, but proofs of areas and volumes did not appear until the arrival of Greek mathematics.

However, although Euclid (c.300 BC) included these proofs in his *Elements*, no definition is given of the terms length, area and volume, but rather they are used implicitly in the diagrams that would later be used. Line, surface and solid are defined thus: a line is a length without width; a surface is that which has only length and width; a solid is that which has length, width and depth. Neither did Euclid define what measuring is. It is a word that he uses not only in relation to these three 'magnitudes', but also in reference to numbers. For example, he defines part and parts in the same way that we would use divisor and non-divisor, but using the word measure: "A number is *part* of a number, the lesser of the greater, when it measures the greater. But *parts* is when it does not measure it." So, for example, 3 is 'part' of 15, and 6 is 'parts' of 15.

We still cannot find a definition for *measure* by other Greek authors, such as Archimedes, who used comparisons with known areas and volumes to find new ones. We can see, for example, that he worked in this way to calculate the volume of a sphere. Such concepts of measurement were enough for the development of mathematics for quite a number of centuries.

Georg Cantor (1845-1918) was the protagonist of the next stage when he gave the first definition of measure in 1883: $m\,(A)$, of an arbitrary set (bounded) $A \subset R^n$. Moreover, Cantor discovered that infinite sets do not always have the same size, that is, the same cardinal. For example, the set of the rational numbers is *numerable*, that is to say, it has the same size as the set of the natural numbers, while that of the real numbers is not. In this sense, measure means to be able to establish a one-to-one correspondence, 'biunivocal' in mathematical terms, between two sets, one of which is $\mathbb{N}$ (natural numbers) or one of its powers ($\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, etc.). In the case of rational numbers, for example, the following relation can be established:

$$\mathbb{Q} \rightarrow \mathbb{N} \times \mathbb{N}$$
$$p/q \rightarrow (p,q)$$

where p/q is the irreducible and unique fraction of any rational number. A similar relation cannot be established between $\mathbb{R}$ and $\mathbb{N}$ or a power of it; so it is said that $\mathbb{R}$ is not countable. There are, therefore, various infinities, some of which are larger than others. Among these infinities there are some that are so large that they have no correspondence with the real three-dimensional space (length, width and height), which is sometimes likened to the vector space $\mathbb{R}^3$.

Other authors, such as Otto Stolz (1842–1905) in 1884 and AdolfVon Harnack (1851–1930) in 1885, give equivalent definitions in $\mathbb{R}$. In them, for example, the rationals of the interval [0,1] measured 1, the same as all the real numbers of [0,1]. We can sense that something is not going to work in this comparison between elements of $\mathbb{Q}$ and elements of $\mathbb{R}$, if we understand that $\mathbb{Q}$ is numerable and $\mathbb{R}$ is not numerable.

The problem lay in managing to differentiate between an enumerable set and a measurable set. The first one is a set that can be put into bijective correspondence with the set of natural numbers; the second is the set that can be put into bijective correspondence with the set of non-negative real numbers. This distinction is the formalisation of the difference between counting and measuring associated with the discrete and the continuous which was introduced in Chapter 1. The English language distinguishes between countable and uncountable objects *(how many / how much)*.

Giuseppe Peano (1858–1932) determined that sets A are *measurable* and gave a definition of their measurement in 1887. He introduced the inner and outer measure of a region R as the least upper bound of all the polygonal regions contained in the interior of R and the greatest lower bound of all the polygonal regions containing this region, respectively. He defined a *measurable* set as the one whose inner measurement coincides with its outer one, and he proved that the measure was additive. He also explained the relation between measure and integration. In 1892, Camille Jordan (1838-1922) gave a simpler definition by using a grid instead of polygons. These definitions are reminiscent of the methods of approximating the number π by means of successive approximations to the perimeter or area of a circle carried out by mathematicians in antiquity (in Egypt, China, India and Greece) who used polygons inscribed and circumscribed to a circle.

Despite the progress made, these definitions were not enough because, for example, rational numbers were not measurable with them. Two years later, Émile

Borel (1871–1956) continued the work on the issue and in his doctoral thesis (1894) established countable additivity for his measurements, which went further than the finite additivity with which Peano had worked, and he gave a definition for sets of zero measure. This new approach led him to find that the rationals of [0,1], which other authors had considered as measuring 1, were a set of zero measure.

From the new concept of measuring established by Borel, Henri Lebesgue (1875–1941), in his doctoral thesis in 1902, was able to develop fundamental concepts for the theory of abstract integration. He extended the Riemann integral – created by Bernhard Riemann (1826-1866), who defined it as the area below a continuous curve – to a new integral, the Lebesque integral, which was also appropriate for discontinuous functions.

To recap, throughout this book we have seen that measurement of the heavens was performed by astronomy; that of the Earth by geodesics; that measurement of time led to the creation of calendars; and the need for the existence of a universal standard of measure led to the metre being established as the standard of length. All these kinds of activities in the physical, astronomical, geodesic, calendrical and metrological worlds would be impossible without the essential contribution of mathematics. Mathematics itself has also developed the concept of measuring through the theory of measurement in what has been named the mathematical model, but a detailed of study of that would require a new book.

Bibliography

ALDER, K., *The Measure of All Things: the Seven-year Odyssey and the Hidden Error that Transformed the World,* New York, Free Press, 2002.

BISHOP, A.J., *Mathematical Acculturalisation. Mathematics Education in a Cultural Perspective*, Reston VA, National Council of Teachers of Mathematics, 2004.

BOURGOING, J., *The Calendar. Measuring Time*, London, Thames & Hudson, 2001.

GUEDJ, D., *The Measure of the World,* Chicago, Chicago University Press, 2001.

KATZ, 5th., MICHALOWICZ, K. (eds.), *Historical Modules for the Teaching and Learning of Mathematics*, Washington, The Mathematical Association of America, 2004.

KUHN, T.S., *The Copernican Revolution: Planetary Astronomy in the Development of Western Thought*, Cambridge MA, Harvard University Press, 1992.

LINDBERG, D.C., *The Beginnings of Western Science*, Chicago, Chicago University Press, 2008.

SOBEL, D., *Longitude: the True Story of a Lone Genius Who Solved the Greatest Scientificc Problem of His Time,* London, HarperCollins, 2005.

Index

Getting the Measure of the World

Calendars, lengths and mathematics

How much time has passed? How high is the moon? How heavy is an elephant? The vital mathematical field of measurement allows us to respond precisely to a multitude of variables that cannot be classified in any other way. Furthermore, the standardisation of measurements has allowed us to speak the same language throughout the world. Without measurement and mathematics, we would live in a very different, almost unimaginable world!